A STUDY OF THE CIRCLE CROSS*

BY

J. L. COOLIDGE

One fundamental characteristic of that discipline which is called by the vague title of "higher geometry" consists in taking as element, not necessarily a point in two or three dimensional space, but any object which can be determined by a finite number of parameters. The student of analytic geometry begins in Cartesian fashion with a point and its coördinates. A little later line and plane coördinates are introduced, and the student feels that the content of his subject has been multiplied by two. Further extensions soon follow, the Plücker line geometry, the coördinates of a circle or sphere, the linear line-complex as space element, the projective geometry of n dimensions. The process is capable of indefinite extension.

In line geometry a very fruitful idea was introduced by Study,† that of taking as element, not a single line, but a pair of lines called a "cross." One line of the pair is finite, the other, in general, at infinity, and each plane through one is perpendicular to the other. The cross takes a more symmetrical shape in non-euclidean space where under the previous definition the two lines are merely mutually polar with regard to the absolute quadric.‡ It is the object of the present paper to extend this concept to circles. *A circle cross will be composed of a pair of circles so related that every sphere through one is orthogonal to the other.* In the first section we shall give a number of preliminary theorems, some well known, concerning the angles of circles. In the second we take systems of circles orthogonal to one sphere. The analogy to the corresponding problem in non-euclidean line geometry is here so close as to be almost identity, and we shall give our results with all possible brevity. In section three we shall discuss circle crosses in general position in space. The simple analytical treatment of the cross as element does not seem possible in this case, but a number of interesting results are obtainable. In the last section we discuss a new transformation which carries circles into circles, but not spheres into spheres. Theorems already familiar are marked with an accent to the number, as Theorem 2′, etc. Others not so marked are believed to be new.

* Presented to the Society December 31, 1912.

† *Geometrie der Dynamen*, Leipzig, 1903.

‡ See, e. g., the author's *Non-Euclidean Geometry*, Oxford, 1909.

§ 1. *The Angles of Circles.*

We shall take as our domain what may be called a *perfect pentaspherical continuum*, i. e., a set of objects called *points* in one-to-one correspondence with all sets of values of five non-simultaneously vanishing coördinates *

$$x_0 : x_1 : x_2 : x_3 : x_4$$

connected by the identical relation

$$\sum_i^{0 \dots 4} x_i^2 \equiv (xx) = 0. \tag{1}$$

A sphere (y) will be given by a relation

$$\sum_i^{0 \dots 4} y_i x_i \equiv (yx) = 0; \tag{2}$$

and we regard the coefficients $y_0, \cdots y_4$ as the coördinates of this sphere. When (y) satisfies the fundamental identity (1) the sphere is said to be *null*, and the point (y) is called its *vertex*. We thus use the same notation for point and sphere coördinates, the point being considered merely as the vertex of a null sphere. The angle of two not null spheres (x) and (y) will be

$$\theta = \cos^{-1} \frac{(xy)}{\sqrt{(xx)}\sqrt{(yy)}}. \tag{3}$$

Every conformal transformation of our space will be given by an orthogonal substitution in our coördinates (x).

Suppose that we have four spheres $(x)\,(y)\,(x')\,(y')$. The coördinates of their common orthogonal spheres are given by the four-rowed determinants $\rho r_s = |\, x_j\, y_k\, x'_l\, y'_m \,|$. This is indeterminate when, and only when, the four spheres are linearly dependent. Assuming that no two of them are identical, this means that some sphere through the circle determined by $(x)\,(y)$ is also a sphere through a circle of $(x')\,(y')$.

THEOREM 1′: *The necessary and sufficient condition that two circles should have more than one common orthogonal sphere is that they should be cospherical.*

THEOREM 2′: *If a circle be cospherical with each of two others which are not cospherical with one another, it is orthogonal to their common orthogonal sphere.*

This second theorem comes at once from the fact that if two spheres be orthogonal to a third, so is their circle of intersection.

Through each not null circle will pass two distinct null spheres. Their vertices are called the *foci* of the circle and serve to determine it: the circle is the locus of points of intersection of isotropic lines through the foci. Every

* The earliest systematic treatment of pentaspherical coördinates was that of DARBOUX, *Sur une classe remarquable de courbes et de surfaces*, 1873.

sphere through the foci of a not null circle cuts it orthogonally, every sphere orthogonal to such a circle passes through its foci. If two circles be so related that the first lies on a sphere through the foci of the second, then a sphere through the second and one focus of the first is orthogonal to the given sphere, and so contains the second focus of the first:

THEOREM 3′: *If two circles be so related that one is orthogonal to a sphere through the other, then the second is orthogonal to a sphere through the first.*

Two such circles are said to be *in involution.*

THEOREM 4′: *The necessary and sufficient condition that two cospherical circles should be in involution is that they should intersect orthogonally.*

The proof given above for theorem 3′ holds when one circle lies on more than one sphere orthogonal to the other, i. e., when it contains the foci of the latter.

THEOREM 5′: *If two circles be so related that two spheres through one are orthogonal to the other, then every sphere through either circle is orthogonal to the other circle and the foci of each lie on the other.*

Two distinct circles which bear the relation of the last theorem are said to be in *bi-involution.** The figure which they form shall be called a *proper cross.* If a circle be null, consisting of two distinct isotropic lines, any circle orthogonal to the pencil of tangent spheres through the null circle will be in bi-involution with the null circle; together they shall be said to form an *improper cross of the first sort.* If a null circle consist of two identical isotropic lines, every sphere through it is null; it will be in bi-involution with itself, and be said to form an *improper cross of the second sort.* The following theorem is immediate.

THEOREM 6: *There is a single circle in bi-involution with a given circle and orthogonal to any given not null sphere which is orthogonal to the given circle.*

THEOREM 7′: *The necessary and sufficient condition that two cospherical not null circles should intersect orthogonally is that they should be in involution.*

THEOREM 8: *A circle cospherical with both circles of a proper cross cuts them both orthogonally.*

This is an immediate result of 2′ and the definition of circles in bi-involution. We proceed to investigate how many circles there are, cospherical and orthogonal to each of two given not null circles. Let the first be given by its foci $(x)\,(y)$ while the second has foci $(x')\,(y')$. A circle cospherical and orthogonal to both shall be determined by the spheres $\lambda\,(x)+\mu\,(y)$, $\lambda'\,(x')+\mu'\,(y')$. Since mutually perpendicular spheres are harmonically separated by the null spheres whose vertices are their foci, our two spheres above are orthogonal

* The term "involution" and "bi-involution" are taken from an admirable article by KOENIGS, *Contributions à la théorie du cercle dans l'espace,*" Annales de la Faculté des Sciences de Toulouse, vol. 2, 1888.

to $\lambda(x) - \mu(y)$, $\lambda'(x') - \mu'(y')$. We thus get the equations

$$\lambda\lambda'(xx') - \mu\mu'(yy') = \lambda\mu'(xy') - \mu\lambda'(yx') = 0.$$

Eliminating $\lambda' : \mu'$, we have

$$\lambda^2(xx')(xy') - \mu^2(xy')(yx') = 0.$$

The discriminant of this equation is

$$4(xx')(yy')(xy')(yx').$$

THEOREM 9: *Two not null and non-cospherical circles are cospherical and orthogonal to two circles of a proper cross, and to no others when, and only when, no focus of one lies on an isotropic with a focus of the other.*

Let us consider these equations in more detail. If

$$(xx') = 0, \qquad (yy')(xy')(yx') \neq 0.$$

The isotropic $(x)(x')$ is the only circle fulfilling the conditions. If

$$(xx') = (yx') = 0, \qquad \mu' = 0.$$

Here one circle passes through one focus of the other, and there are ∞^1 null circles fulfilling the conditions. If

$$(xx') = (yy') = 0,$$

each focus of one is on an isotropic with one focus of the other, and the two circles meet two common generators of their common orthogonal sphere. These two generators must belong to the same system, as otherwise the circles would be tangent and cospherical; the circles are said in this case to be *paratactic.** They are invariant under a one parameter continuous group of spherical transformations which leave invariant these two isotropic lines, as well as all those of the other system on the common orthogonal sphere. Paratactic circles will exist even in the limiting case where this sphere is null, as we see by inverting two paratactic lines of non-euclidean space. If

$$(xx') = (yy') = (yx') = 0, \qquad (xy') \neq 0,$$

we have a case essentially like the last but one. If

$$(xx') = (xy') = (yx') = (yy') = 0.$$

The circles are in bi-involution and have ∞^2 common cospherical and orthogonal circles.

* The concept of paratactic circles appears to be new, although coming directly from that of paratactic or Clifford parallel lines of non-euclidean space. In fact the whole question of circles cospherical with and orthogonal to two given circles seems never to have been adequately discussed.

THEOREM 10: *The necessary and sufficient condition that two circles should have but one common cospherical and orthogonal circle is that they should be non-cospherical, and that a single focus of one should be on an isotropic with a single focus of the other.*

THEOREM 11: *The necessary and sufficient condition that two circles not null or cospherical should be cospherical and orthogonal to a singly infinite system of circles is that they should be paratactic, in which case these circles are not null, or that one focus of the one should be on an isotropic with each focus of the other, when the ∞^1 circles are all null.*

THEOREM 12: *The necessary and sufficient condition that two not null circles should be cospherical and orthogonal to the same two parameter family of circles is that they should be in bi-involution.*

Two circles are said to be in *general position* when the only circles cospherical and orthogonal to them are those of a proper cross. We shall define as the *angles* of two circles in general position those formed by the pairs of spheres through them, and through the two circles cospherical and orthogonal to them. These angles have definite values in the general case, and in the case of paratactic circles, otherwise they are null or indeterminate. Reverting to our previous notations, we have as expressions for our angles

$$(4)\quad \cos^2\theta_1+\cos^2\theta_2=\frac{2[(xx')\,(yy')+(yx')\,(xy')]}{(xy)\,(x'y')},\ \cos^2\theta_1\cos^2\theta_2=\frac{\begin{vmatrix}(xx')&(xy')\\(yx')&(yy')\end{vmatrix}^2}{(xy)^2\,(x'y')^2}.$$

THEOREM 13: *The necessary and sufficient condition that two circles should be paratactic is that their angles should be equal, but not null or indeterminate.*

Suppose that we have two paratactic circles, and a third circle cospherical with both. We assume that the common orthogonal sphere is not null. The one parameter continuous group under which the paratactic circles are invariant will carry the third circle into a series of circles cospherical with the original two. All circles of this series must intersect the same two generators of the other system of the common orthogonal sphere, since all of these generators are invariant under the group; hence all circles of the series are paratactic. They will determine a second one parameter group, and series of circles containing, among others, the original two paratactic circles. This same configuration will hold even when the sphere is null, as we see by continuity, or by inverting the circles into lines. Remembering, lastly, that the only surfaces which have a double generation by means of circles, where each circle of one generation is cospherical with each of the other, are cyclides, we have

THEOREM 14: *Two paratactic circles belong to a singly infinite system of cyclides each having two conjugate generations made up of paratactic circles.*

§ 2. *Circles Orthogonal to One Sphere.**

Let us next turn our attention particularly to those circles which are all orthogonal to one not null sphere; those orthogonal to a null sphere are obtained by inversion from straight lines. We may freely assume that the fundamental sphere is $x_5 = 0$. We determine each circle of the system then by its six Plücker coördinates

$$\rho p_{0i} = x_0 y_i - x_i y_0, \qquad \rho p_{jk} = x_j y_k - x_k y_j \quad (j \neq 0,\ k \neq 0).$$

It is clear that these coördinates, being supposed homogeneous, are unaltered if (x) and (y) be replaced by any two of their linear combinations so that our coördinates depend upon the circle and not on the two particular spheres through it. Two circles of our system will be cospherical when they have a common orthogonal sphere other than the fundamental one; the necessary and sufficient condition for this is

$$(5) \quad p_{01} q_{23} + p_{02} q_{31} + p_{03} q_{12} + p_{23} q_{01} + p_{31} q_{02} + p_{12} q_{03} \equiv (p/q) = 0.$$

We have also the fundamental identity

$$(6) \quad (p/p) = 0.$$

Conversely we prove exactly as in the Plücker line geometry that every set of homogeneous coördinates which satisfy this last identity will correspond to a single circle orthogonal to our fundamental sphere.

There is a perfect one-to-one correspondence between the circles orthogonal to a not null sphere and the lines of projective space; cospherical circles corresponding to intersecting lines. A sphere orthogonal to the fundamental sphere will correspond to a point, a pair of points mutually inverse in the fundamental sphere will correspond to a plane. A pencil of cospherical circles will correspond to a pencil of lines, a cyclide anallagmatic in the fundamental sphere will correspond to a quadric.

When the fundamental sphere is not a plane, and the figures are looked upon as in euclidean space, this correspondence may be easily effected by referring each circle to its axis, i. e., the line through its center, orthogonal to its plane. Two circles (p) and (q) will be in involution if

$$(7) \quad \Sigma p_{ij} q_{ij} = 0.$$

If two circles of our system be in involution, each is cospherical with that circle which is orthogonal to the fundamental sphere, and in bi-involution with the other, and conversely. The circle of our system in bi-involution

* For an analytic treatment of this topic see the dissertation of FORBES: *The Geometry of Circles Orthogonal to a Given Sphere*, New York, 1904. Synthetically by C. L. E. MOORE: *Circles Orthogonal to One Sphere*, Annals of Mathematics, Series 2, vol. 8, 1907.

with (p) will thus have the coördinates

$$p'_{0i} = \rho p_{jk}, \qquad p'_{jk} = \rho p_{0i}. \tag{8}$$

The necessary and sufficient condition that two circles of our system should be cospherical and orthogonal is

$$(p/q) = \Sigma p_{ij}\, q_{ij} = 0.$$

The only circles in involution with themselves are null circles; the condition here is

$$\Sigma p_{ij}^2 = 0.$$

The *fundamental invariant* of two not null circles is the product of the cosines of their angles;* this, by (4), has the value

$$I(p\,q) \equiv \cos\theta_1 \cos\theta_2 = \frac{\Sigma p_{ij}\, q_{ij}}{\sqrt{\Sigma p_{ij}^2}\sqrt{\Sigma q_{ij}^2}}. \tag{9}$$

The necessary condition that two circles should be paratactic is

$$\left\{[(p/q)+\Sigma p_{ij}\, q_{ij}]^2 - \Sigma p_{ij}^2\, \Sigma q_{ij}^2\right\}\left\{[(p/q)-\Sigma p_{ij}\, q_{ij}]^2 - \Sigma p_{ij}^2\, \Sigma q_{ij}^2\right\} = 0. \tag{10}$$

A system of circles orthogonal to our fundamental sphere and satisfying an equation of the type

$$\Sigma a_{ij}\, p_{ij} = 0 \tag{11}$$

is called a "*linear* complex." If

$$(a/a) = 0$$

the complex is *special*, and is composed of the assemblage of all circles cospherical with a given circle, if not we have a rather more complicated system. The circles orthogonal to our fundamental sphere, and in bi-involution with those of our complex (11) belong to a second linear complex, whose equation is

$$(a/p) = 0. \tag{12}$$

The congruence of circles common to these two (in the general case where equations (11) and (12) are not equivalent to one another) are all cospherical with the two circles

$$\lambda a_{0i} + \mu a_{jk},$$

$$(\lambda^2 + \mu^2)(a/a) + 2\lambda\mu\Sigma a_{ij}^2 = 0.$$

Our linear complex shall be said to be *general* when these two are distinct. We call them (α) and (α') and note that they are in bi-involution, since

* This invariant is due to KOENIGS, loc. cit., p. 79, i. e., he gives the same analytic form, but a different interpretation. A neat interpretation as the cosine of a single angle has recently been found by Mr. David Barrow of Harvard University.

equations (11) and (12) are merely interchanged when each Plücker coördinate is replaced by its complementary. We also see that we may write

$$\alpha_{ij} = \lambda a_{ij} + \mu a_{kl}, \qquad \alpha'_{ij} = \lambda a_{kl} + \mu a_{ij},$$

$$\frac{I(\alpha' p)}{I(\alpha p)} = \frac{\lambda (a/p) + \mu \Sigma a_{ij} p_{ij}}{\lambda \Sigma a_{ij} p_{ij} + \mu (a/p)}.$$

If, therefore, (p) be a line of our linear complex, we have

$$\frac{I(\alpha' p)}{I(\alpha p)} = \frac{\lambda}{\mu} = -\frac{\Sigma a_{ij}^2 + \sqrt{(\Sigma a_{ij}^2)^2 - (a/a)^2}}{(a/a)}.$$

THEOREM 15: *The general linear complex of circles orthogonal to a not null sphere is composed of all circles whose fundamental invariants with the two circles of a proper cross orthogonal to the sphere bear to one another a fixed ratio whose square is not unity and, conversely, every such assemblage is a linear complex.*

The circles (α) and (α') are called the *axial circles* of the complex, our theorem will hold even when the fundamental sphere is null, as we see by continuity, or by inversion from the non-euclidean line geometry.*

We shall confine our discussion in the present section to the general linear complex and not enter into any special case except the following. If

$$a_{0i} = \rho a_{jk}, \qquad a_{jk} = \rho a_{0i}, \qquad \rho^2 = 1,$$

and if we put

$$q'_{ij} = (a/a) q_{ij} - 2 (\Sigma a_{ij} q_{ij}) a_{ij},$$

where (q) is any circle orthogonal to the fundamental sphere, but not belonging to our complex, then

$$(a/a) = \Sigma a_{ij}^2, \qquad \Sigma q'^2_{ij} = (a/a)^2 \Sigma q_{ij}^2,$$

$$\frac{I(q'/p)}{I(q/p)} = 1.$$

If we choose our coördinate system so that the axial circles are $x_0 = x_1 = 0$; $x_2 = x_3 = 0$ our complex becomes

$$a_{01} p_{01} + a_{23} p_{23}.$$

If a general linear complex be given, the axial circles are given, the converse is not true. We may, in fact, replace our complex (a) by (a'), where

$$a'_{ij} = \lambda a_{ij} + \mu a_{kl}.$$

The equations to determine the axial circles are replaced by simple linear combinations of themselves, and the axial cross is as before. We are thus led

* See the author: *The Metrical Aspect of the Line-Sphere Transformation*, these Transactions, vol. 12, 1911.

to a simple set of coördinates for this cross *

$$a_{0i} + a_{jk} = \rho X_i, \qquad a_{0i} - a_{jk} = \sigma \dot{X}_i.$$

If a general linear complex be given, the triads of separately homogeneous coördinates are immediately found, and it is easily seen that they are merely affected by constant multipliers if the complex be replaced by another with the same axial circles. Conversely, when we have the two sets of separately homogeneous coördinates $(X)\,(\dot{X})$ with the limitation

$$(\Sigma X_i^2)\,(\Sigma \dot{X}_i^2) \neq 0,$$

the linear complex

$$2a_{0i} = X_i + \dot{X}_i, \qquad 2a_{jk} = X_i - \dot{X}_i$$

will be a general complex, whose axial cross has the coördinate values $(X)\,(\dot{X})$.

THEOREM 16: *The assemblage of all proper crosses orthogonal to a not null sphere may be put into one to one correspondence with the assemblage of all pairs of points of two projective planes not lying on two specified conics, one in each plane.*

We may, with a little ingenuity, extend our correspondence so as to include points of these exiled conics, we shall not however pause to do so here. If two circles have the coördinates (a) and (b), we shall find

$$\cos\theta_1 \cos\theta_2 = \frac{\Sigma a_{ij}\, b_{ij}}{\sqrt{\Sigma a_{ij}^2}\sqrt{\Sigma b_{ij}^2}}, \qquad \sin\theta_1 \sin\theta_2 = \frac{(a/b)}{\sqrt{\Sigma a_{ij}^2}\sqrt{\Sigma b_{ij}^2}},$$

$$\cos(\theta_1 \pm \theta_2) = \frac{\Sigma X_i X_i'}{\sqrt{\Sigma X_i^2}\sqrt{\Sigma X_i'^2}}, \qquad \cos(\theta_1 \mp \theta_2) = \frac{\Sigma \dot{X}_i \dot{X}_i'}{\sqrt{\Sigma X_i^2}\sqrt{\Sigma X_i'^2}}.$$

The condition that two proper circles shall be cospherical, or that each should be cospherical with the circle of our system in bi-involution with the other is

$$\frac{\Sigma X_i X_i'}{\sqrt{\Sigma X_i^2}\sqrt{\Sigma X_i'^2}} = \pm \frac{\Sigma \dot{X}_i \dot{X}_i}{\sqrt{\Sigma \dot{X}_i^2}\sqrt{\Sigma \dot{X}_i'^2}}.$$

Two circles orthogonal to our fundamental sphere are cospherical and orthogonal if

$$\Sigma X_i X_i' = \Sigma \dot{X}_i \dot{X}_i' = 0. \tag{13}$$

It will be convenient to say in the present case that the two crosses intersect orthogonally. The necessary conditions for parataxy become

$$\left[\Sigma \begin{vmatrix} X_i & X_i' \\ X_j & X_j' \end{vmatrix}^2\right]\left[\Sigma \begin{vmatrix} \dot{X}_i & \dot{X}_i' \\ \dot{X}_j & \dot{X}_j' \end{vmatrix}^2\right] = 0.$$

* See the author's *Non-Euclidean Geometry*, cit. Ch. XII. The developments of the present section follow so closely those previously established that we give them here in all brevity.

The necessary and sufficient conditions are thus

$$\rho X_i' = X_i \quad \text{or} \quad \sigma \dot{X}_i' = \dot{X}_i .$$

The simplest one parameter families of circle crosses are those whose coördinates are linearly dependent upon those of two fixed crosses,

$$X_i = \lambda Y_i + \mu Z_j , \qquad \dot{X}_i = \lambda \dot{Y}_i + \mu \dot{Z}_i . \tag{14}$$

These are the axial complexes of the pencil of linear complexes

$$a_{ij} = \lambda b_{ij} + \mu c_{ij} .$$

These families fall under one of two decidedly different types.

$$\rho Y_i \equiv Z_i , \qquad \sigma \dot{Y}_i \not\equiv \dot{Z}_i . \tag{a}$$

Here all circles of the series are paratactic. Let us determine $(\dot{X}')$ from the equations

$$\Sigma \dot{X}_i' \dot{Y}_i = \Sigma \dot{X}_i' \dot{Z}_i = 0 ,$$

then

$$\Sigma \dot{X}_i' \dot{X}_i = 0 .$$

On the other hand if (X') merely satisfy the equations

$$\Sigma Y_i X_i' = 0$$

the crosses $(\dot{X}')$ $(\dot{X}')$ generate a similar system, and every cross (X) $(\dot{X})$ cuts every cross (X') $(\dot{X}')$ orthogonally. The circles of our series generate a cyclide.

$$\rho Y_i \not\equiv Z_i , \qquad \sigma \dot{Y}_i \not\equiv \dot{Z}_i ,$$

$$U_i = Y_j Z_k - Y_k Z_j , \qquad \dot{U}_i = \dot{Y}_i \dot{Z}_k - \dot{Y}_k \dot{Z}_j , \tag{b}$$

$$\Sigma U_i X_i = \Sigma \dot{U}_i \dot{X}_i = 0 .$$

All crosses of our series cut one fixed cross orthogonally. The surface so generated, even in degenerate cases, shall be a called a *pseudo-cylindroid.** It has one general and very fundamental property. Suppose that we have an analytic congruence of circles orthogonal to our fundamental sphere

$$X_i = X_i (uv) , \qquad \dot{X}_i = \dot{X}_i (uv) .$$

Let us further assume

$$\left| X \frac{\partial X}{\partial u} \frac{\partial X}{\partial v} \right| \not\equiv 0 , \qquad \left| \dot{X} \frac{\partial \dot{X}}{\partial u} \frac{\partial \dot{X}}{\partial v} \right| \not\equiv 0 .$$

This means that our circles can not all be assembled into one parameter families of paratactic circles. We mean by a circle *in general position* in our congruence one whose coördinates will not make the left sides of the above

* From the analogous figure in line-geometry, the euclidean or non-euclidean cylindroid.

expressions vanish. We have

THEOREM 17: *If an analytic congruence of circles be given orthogonal to one sphere, whose members can not be all assembled into one parameter paratactic families, and if a circle be taken in general position in such a congruence, then the circles cospherical and orthogonal to this and to the infinitely near circles of the congruence will generate a pseudo-cylindroid.*

It is time to hunt for a canonical form for our pseudo-cylindroid. We may take for the cross which all cut orthogonally

$$(1\,0\,0)\,(1\,0\,0).$$

Two crosses of the series will intersect orthogonally, we have but to solve the equations

$$\lambda\lambda'\,\Sigma Y_i\,Y_i + (\lambda\mu' + \mu\lambda')\,\Sigma Y_i\,Z_i + \mu\mu'\,\Sigma Z_i\,Z_i$$
$$= \lambda\lambda'\,\Sigma \dot{Y}_i\,\dot{Y}_i + (\lambda\mu' + \mu\lambda')\,\Sigma \dot{Y}_i\,\dot{Z}_i + \mu\mu'\,\Sigma \dot{Z}_i\,\dot{Z}_i = 0.$$

Eliminating $\lambda' : \mu'$, we get a quadratic in $\lambda : \mu$; its discriminant is the simultaneous invariant of

$$\Sigma X_i^2 = 0, \qquad \Sigma \dot{X}_i^2 = 0,$$

when these are written as quadratics in $\lambda : \mu$. When our crosses have real coördinates the roots of each of these two quadratic equations are conjugate imaginary pairs of values, and two such pairs can not separate one another harmonically; hence the simultaneous invariant can not vanish, and there are usually two distinct crosses of the series cutting one another orthogonally. Taking these for $(0\,1\,0)\,(0\,1\,0)$ and $(0\,0\,1)\,(0\,0\,1)$ we may reduce the equations of our series to

$$\rho X_1 = \alpha \dot{X}_1 = 0, \qquad \rho X_2 = \beta \dot{X}_2, \qquad \rho X_3 = \gamma \dot{X}_3 \quad (\alpha, \beta, \gamma, \text{const.}).$$

To find the point equation of the surface let (p) be the Plücker coördinates of a circle of one of our crosses;

$$p_{01} = p_{23} = 0,$$

$$\beta\,(p_{02} - p_{31})\,(p_{03} + p_{12}) = \gamma\,(p_{03} - p_{12})\,(p_{02} + p_{31}).$$

A point (x) will be on this circle if

$$x_2\,p_{02} + x_3\,p_{03} = x_0\,p_{02} + x_1\,p_{i2} = 0.$$

Eliminating from these equations and the identity (6) we have

$$\beta\,(x_0\,x_2 + x_3\,x_1)\,(x_0\,x_3 + x_1\,x_2) + \gamma\,(x_3\,x_1 - x_0\,x_2)\,(x_1\,x_2 - x_0\,x_3) = 0.$$

THEOREM 18: *The non-degenerate pseudo-cylindroid is a surface of the eighth order.*

Among congruences of crosses the most interesting is the *chain congruence* which consists of crosses whose coördinates are linearly dependent on three of their number, or the axial crosses of a net of linear complexes. We write in this case

$$X_i = aY_i + bZ_i + cT_i, \qquad \dot{X}_i = a\dot{Y}_i + b\dot{Z}_i + c\dot{T}_i. \tag{15}$$

Various sub-cases must be noticed.

$$(a) \qquad |YZT| = |\dot{Y}\dot{Z}\dot{T}| = 0.$$

Assuming that we have surely a two parameter family of crosses, the first minors can not all vanish in either of these determinants. We have the congruence of crosses cutting a given cross orthogonally,

$$(b) \qquad |YZT| = 0, \qquad |\dot{Y}\dot{Z}\dot{T}| \neq 0,$$

$$T_i = \lambda Y_i + \mu Z_i, \qquad X_i = (a + c\lambda)\, Y_i + (b + c\mu)\, Z_i.$$

If

$$\frac{a + c\lambda}{b + c\mu} = \frac{p}{q} = \text{const.}$$

$$c = \frac{bp - aq}{q\lambda - p\mu},$$

$$X_i = pY_i + qZ_i, \qquad \dot{X}_i = a\left[\dot{Y}_i - \frac{q\dot{T}_i}{q\lambda - p\mu}\right] + b\left[Z_i + \frac{p\dot{T}_i}{q\lambda - p\mu}\right].$$

The crosses may be grouped into paratactic generations of ∞^1 cyclides.

$$(c) \qquad |YZT| \cdot |\dot{Y}\dot{Z}\dot{T}| \neq 0.$$

Solving the equations (15) for a, b, c, we have

$$\rho\dot{X}_i = \Sigma_j a_{ij} X_j. \tag{16}$$

The crosses intersecting orthogonally pairs of crosses of our congruence will generate the *conjugate congruence* which is of like structure,

$$\sigma U_i = \Sigma_j A_{ij} U_j. \tag{17}$$

The relation between these two congruences is reciprocal, each is made up of the crosses intersecting orthogonally pairs of crosses of the other. Let us see if we can find any crosses which belong simultaneously to the two conjugate congruences. For such an one

$$\tau X_i = \Sigma_j b_{ij} X_j.$$

These equations will yield a cubic equation in τ. Let us say that the chain congruence is *general* when this equation has three distinct roots. Each root gives a cross common to the two congruences, the common perpendicular

to each two of these crosses must be the third. Each two must intersect orthogonally. We may then take them as the basis of our coördinate system, namely the crosses

$$(1\,0\,0)(1\,0\,0), \qquad (0\,1\,0)(0\,1\,0), \qquad (0\,0\,1)(0\,0\,1).$$

The general chain congruence and its conjugate will then appear in the canonical form

$$\rho\dot{X}_i = a_i X_i, \qquad \sigma U_i = a_i \dot{U}_i,$$
$$a_1 a_2 a_3 (a_1 - a_2)(a_2 - a_3)(a_3 - a_1) \neq 0. \tag{18}$$

The condition that two crosses of our conjugate congruences should be cospherical or that one should be cospherical with the circle in bi-involution with the other is

$$\Sigma U_i X_i \left[\sqrt{\Sigma a_i X_i^2}\sqrt{\Sigma \frac{1}{a_i}\cdot U_i^2} \pm \sqrt{\Sigma X_i^2}\sqrt{\Sigma U_i^2} \right] = 0.$$

If the first factor vanish, then the cospherical circles cut orthogonally, as do their mates. If the second factor vanish, one circle of each cross of the first congruence for which

$$q\Sigma a_i X_i^2 = p\Sigma X_i^2$$

is cospherical with one circle of each cross of the second congruence for which

$$p\Sigma \frac{1}{a_i}\cdot U_i^2 = q\Sigma U_i^2.$$

THEOREM 19: *The circles of a general chain congruence cospherical with a circle of the conjugate congruence and orthogonal to it generate a pseudo-cylindroid, those which are cospherical but not orthogonal to such a circle generate a cyclide; the conjugate generation of this cyclide belongs to the conjugate congruence.*

If a cyclide have our fundamental sphere as one of its spheres of inversion, it is clear that the circles of either corresponding generation can be rationally expressed in the form

$$X_i = t^2 Y_i + 2tZ_i + T_i, \qquad \dot{X}_i = t^2 \dot{Y}_i + 2t\dot{Z}_i + \dot{T}_i.$$

THEOREM 20: *The circles cospherical and orthogonal to pairs of circles of one generation of a cyclide generate a chain congruence which includes the conjugate generation of the same cyclide.*

Taking the equation of our general chain congruence in the form (16) we may assume that the sphere $x_0 = 0$ bears no special relation to the congruence. All circles of this sphere satisfy the relation

$$\dot{X}_i = \rho X_i.$$

Substituting in (16) we have a cubic in ρ.

THEOREM 21: *Three circles of a general chain congruence lie on an arbitrary sphere orthogonal to the fundamental sphere, and three also of the conjugate congruence. The triangles formed by the first three circles are polars to those formed by the second three.*

Our congruence is made up of pairs of circles in bi-involution. If a circle lie on a sphere orthogonal to the fundamental sphere, its mate passes through the foci of the circle common to these two spheres.

THEOREM 22: *Through each pair of points which are mutually inverse with regard to the fundamental sphere will pass three circles of a general chain congruence, and three of the conjugate congruence. Each circle of one triad cuts two of the other orthogonally.*

§ 3. *Circles in Space.*

We may determine a circle in general position in pentaspherical space by Plücker coördinates exactly as we did in the special case when it was orthogonal to a given sphere, albeit the formulæ are somewhat more complicated in the general case. If our circle be given by the spheres (x) and (y), we shall write

$$\rho p_{ij} = x_i y_j - x_j y_i. \tag{19}$$

Of these coördinates there are ten which are essentially distinct, and when we use the summation sign we shall mean to cover ten only. For some purposes it is better to let both i and j take any one of the five values, writing the additional equations

$$p_{ij} = - p_{ji}, \qquad p_{ii} = 0. \tag{20}$$

It is evident that as the assemblage of all circles depends upon but six parameters the Plücker coördinates must be connected by certain identical relations, and this is the case, for there are five fundamental identities, which we write at length, largely to establish a convention of sign.

$$\begin{aligned}
\tfrac{1}{2}\Omega_0 (pp) &= p_{12} p_{34} + p_{13} p_{42} + p_{14} p_{23} = 0, \\
\tfrac{1}{2}\Omega_1 (pp) &= p_{02} p_{43} + p_{04} p_{32} + p_{03} p_{24} = 0, \\
\tfrac{1}{2}\Omega_2 (pp) &= p_{01} p_{34} + p_{03} p_{41} + p_{04} p_{13} = 0, \\
\tfrac{1}{2}\Omega_3 (pp) &= p_{01} p_{42} + p_{04} p_{21} + p_{02} p_{14} = 0, \\
\tfrac{1}{2}\Omega_4 (pp) &= p_{01} p_{23} + p_{02} p_{31} + p_{03} p_{12} = 0.
\end{aligned} \tag{21}$$

The polarized form for $\Omega_i (pp)$ shall be written $\Omega_i (pq)$.* It is clear that these identities can not all be independent as then we should have but ∞^4 circles. In fact if (p) be any system of homogeneous values satisfying

* The credit for introducing these coördinates for a circle seems to be due to STEPHANOS. See his note *Sur une configuration remarquable de quinze cercles dans l'espace*, Comptes Rendus, Vol. 93 (1882).

merely (20) we shall have

$$\sum_n p_{in}\,\Omega_n\,(pp) \equiv 0. \tag{22}$$

If $p_{ij} \neq 0$, every set of values which satisfy

$$\Omega_k\,(pp) = \Omega_l\,(pp) = \Omega_m\,(pp) = 0$$

will also satisfy

$$\Omega_i\,(pp) = \Omega_j\,(pp) = 0.$$

Every circle will thus have ten homogeneous and distinct coördinates satisfying our equations (21). Conversely if any set of homogeneous values, not all zero be given which satisfy (20) and (21), they will be the Plücker coördinates of just one circle. The proof is straightforward, and perfectly well known.*

It is occasionally advantageous to determine a circle, not by two spheres through it, but by three spheres to which it is orthogonal. Let these be the spheres $(r)\,(s)\,(t)$. If (u) and (v) be arbitrary spheres, we may imagine that our circle lies on the spheres (x) and (y), where

$$x_i = |\,r_j\,s_k\,t_l\,u_m\,|, \qquad y_i = |\,r_j\,s_k\,t_l\,v_m\,|;$$

substituting in (19), we get by a well-known theorem in determinants

$$p_{ij} = \Delta \begin{vmatrix} r_k & r_l & r_m \\ s_k & s_l & s_m \\ t_k & t_l & t_m \end{vmatrix}. \tag{23}$$

The condition that the two circles (p) and (q) be in involution is easily found to be

$$\Sigma p_{ij}\,q_{ij} = 0. \tag{24}$$

We get from the invariant equation (9),

$$I\,(p, q) = \frac{\Sigma p_{ij}\,q_{ij}}{\sqrt{\Sigma p_{ij}^2}\,\sqrt{\Sigma q_{ij}^2}}. \tag{25}$$

Two circles will be cospherical if

$$\Omega_i\,(pq) \equiv 0. \tag{26}$$

The common orthogonal sphere to two non-cospherical circles (p) and (q) is

$$\rho r_i = \Omega_i\,(pq). \tag{27}$$

Two spheres will have a common point if this sphere be null, i. e., if

$$\Sigma\Omega_i^2\,(pq) = 0. \tag{28}$$

If (x) and (y) be the foci of (p), and (q) be in bi-involution therewith, we may

* See, e. g., KOENIGS, loc. cit.

write

$$q_{ij} = \begin{vmatrix} x_k & x_l & x_m \\ y_k & y_l & y_m \\ t_k & t_l & t_m \end{vmatrix}.$$

The condition that two circles be in bi-involution is thus

$$\Sigma_n p_{in} q_{in} \equiv 0. \tag{29}$$

The necessary condition that two circles be paratactic is

$$[\Sigma p_{ij}^2 \Sigma q_{ij}^2 + (\Sigma p_{ij} q_{ii})^2 - \Sigma \Omega_i^2 (pq)]^2 = 4 (\Sigma p_{ij} \Sigma_{ij})^2 \Sigma p_{ij}^2 \Sigma q_{ij}^2.$$

We see, in fact, that this is an invariant condition for all quinary orthogonal substitutions, and it reduces to (11) when the common orthogonal sphere is $x_4 = 0$.

A five parameter family of circles shall be called a *hypercomplex*. The simplest type of hypercomplex is the linear one, given by an equation of the form

$$\Sigma a_{ij} p_{ij} = 0. \tag{30}$$

As the sphere $x_4 = 0$ may be supposed to bear no special relation to this hypercomplex, we have

THEOREM 23′. *The circles of a linear hypercomplex orthogonal to an arbitrary sphere generate a linear complex.*

THEOREM 24′. *The assemblage of all circles in involution with a given circle is a linear hypercomplex.*

The condition that our hypercomplex (30) should be of this latter type is manifestly

$$\Omega_i (aa) \equiv 0.$$

Let us now suppose that this condition is not fulfilled. Our hypercomplex will have two fundamental invariants of Koenigs which we write

$$\Sigma a_{ij}^2 = I, \qquad \Sigma \Omega_i^2 (aa) = J.$$

If (30) be rewritten in the form

$$\Sigma a_{ij} (x_i y_j - x_j y_i) = 0$$

then if (y) be considered known, the sphere (x) will be orthogonal to the sphere

$$z_i = \sum_n a_{in} y_n.$$

It will be found immediately that

$$\Sigma z_n \Omega_n (aa) = (yz) = 0.$$

The circles of the hypercomplex on an arbitrary sphere are orthogonal to its

intersection with a sphere which is orthogonal to it and to a fixed sphere

$$r_i = \Omega_i (aa).$$

This latter is called the *central sphere* of the hypercomplex. We notice that (z) disappears when (y) takes the value $\Omega(aa)$ so that all circles of the central sphere belong to the hypercomplex. If $J \neq 0$ we may take as the central sphere of the hypercomplex $x_4 = 0$. We have then

$$a_{i4} \equiv 0.$$

The equation of the hypercomplex becomes then identical with (11); we may find (α) and (α') as in the last section, and we have the fundamental equations

$$(31) \qquad \frac{I(\alpha' p)}{I(ap)} = -\frac{\Sigma a_{ij} + \sqrt{(\Sigma a_{ij})^2 - (a/a)^2}}{(a/a)} = \frac{-I + \sqrt{I^2 - J}}{\sqrt{J}}.$$

We define as *general* any linear hypercomplex for which

$$J(I^2 - J) \neq 0.$$

THEOREM 25: *The general linear hypercomplex consists in the assemblage of all circles whose fundamental invariants with the two circles of a proper cross bear to one another a fixed ratio whose square is not unity. The common orthogonal sphere to the two circles of the cross is the central sphere of the hypercomplex.**

We find for our general linear hypercomplex the canonical form

$$(32) \qquad [I + \sqrt{I^2 - J}]\, p_{0i} + \sqrt{J} p_{23} = 0.$$

Let us establish this fundamental theorem in another way, without altering the coördinate system to fit the given hypercomplex. Assuming always that our hypercomplex is general we introduce some new symbols as follows:

$$(33) \qquad A_{ij} = \Omega_k (aa)\, a_{lm} + \Omega_l (aa)\, a_{mk} + \Omega_m (aa)\, a_{kl}.$$

We easily reach the following identities with the aid of (22):

$$(34) \qquad \Omega_i (aA) = I\Omega_i (aa), \qquad \Omega_i (AA) = J\Omega_i (aa),$$

$$\Sigma a_{ij} A_{ij} = J, \qquad \Sigma A_{ij}^2 = IJ.$$

If, then, we write

$$\alpha_{ij} = x a_{ij} + A_{ij},$$

$$\Omega_i (\alpha\alpha) = (x^2 + 2xI + J)\, \Omega_i (aa).$$

Let x_1 and x_2 be the roots of the equation

$$(35) \qquad x^2 + 2xI + J = 0.$$

* This fundamental property appears to have escaped the notice of previous writers dealing with the linear hypercomplex.

We have two circles

$$\alpha_{ij} = x_1 a_{ij} + A_{ij}, \qquad \alpha'_{ij} = x_2 a_{ij} + A_{ij}$$
$$\Sigma\alpha_{ij}^2 = 2x_1 (J - I^2) \qquad \Sigma\alpha_{ij}'^2 = 2x_2 (J - I^2).$$

For a circle of our given hypercomplex we shall have

$$\frac{I(\alpha' p)}{I(\alpha p)} = \frac{\sqrt{x_1}}{\sqrt{x_2}} - \frac{-I + \sqrt{I^2 - J}}{\sqrt{J}}. \tag{31}$$

Notice also that

$$\sum_n \alpha_{in}\, \Omega_n (aa) = \sum_n A_{in}\, \Omega_n (aa)$$
$$= \sum_{jk} (a_{lm} + a_{ml})\, \Omega_j\, \Omega_k$$
$$= 0;$$

the two circles (α) (α') are orthogonal to the central sphere. We have also

$$\sum_n \alpha_{in} \alpha'_{in} = 0.$$

This shows that they are in bi-involution. Our theorem is established again. We shall call these the *axial circles*, or circles of the *axial cross* of the hypercomplex. We may find ∞^1 other linear hypercomplexes with the same axial cross; we have but to write

$$a'_{ij} = pa_{ij} + \frac{qA_{ij}}{\sqrt{J}},$$

$$\Omega_i (a'\, a') = \left(p^2 + 2pq\,\frac{I}{\sqrt{J}} + q^2\right) \Omega_i (aa),$$

$$\sqrt{J'} = \left(p^2 + 2pg\,\frac{I}{\sqrt{J}} + q^2\right) \sqrt{J},$$

$$A'_{ij} = \left(p^2 + 2pq\,\frac{I}{\sqrt{J}} + q^2\right) [pA_{ij} + q\sqrt{J} a_{ij}].$$

The two circles whose coördinates are linear combinations of a_{ij} and A_{ij} may be expressed equally well by means of a linear combination of a'_{ij} and A'_{ij}.

$$\left(a_{ij} + \frac{A_{ij}\,\delta}{\sqrt{J'}}\right) = (p + q\delta) \left(a_{ij} + \frac{A_{ij}\,\delta}{\sqrt{J}}\right),$$
$$\delta^2 = 1.$$

If, then, we introduce the complex variable

$$(x + y\delta), \qquad \delta^2 = 1,$$

we may thus take the expressions

$$\left(a_{ij} + \frac{A_{ij}\,\delta}{\sqrt{J}}\right)$$

as homogeneous complex coördinates of our cross, for if we multiply them by any complex multiplier of the present type, the linear hypercomplex is replaced by another with the same axial cross. Conversely if a proper cross be given, there are ∞^1 linear hypercomplexes having it as axial cross (we have but to consider the case where the orthogonal sphere is $x_4 = 0$) and we may find for it a set of homogeneous complex coördinates of the present type. The only drawback to these coördinates is that (a) and (A) are not independent variables.

We must now see what becomes of the axial cross of our linear hypercomplex when the latter takes certain special forms. Let us first multiply our complex coördinates by $\sqrt{J}$, we get $\sqrt{J}a_{ij} + A_{ij}\,\delta$. Assuming then that

$$\sqrt{J} = 0, \qquad J^2 - I \neq 0.$$

The central sphere is null, equation (35) has one root $x = 0$ giving the circle (A) which by (34) is null; the other circle of the cross is

$$a'_{ij} = -2Ia_{ij} + A_{ij},$$

and not null; we have an improper cross of the first sort. Suppose next

$$I^2 - J = 0.$$

The two members of our cross fall together, so that we have an improper cross of the second sort imbedded in the central sphere, which is null if $J = 0$.

Suppose thirdly that

$$A_{ij} = \rho a_{ij}, \qquad \Omega_i(aa) \not\equiv 0.$$

The equation to determine the axial cross becomes entirely illusory, we have exactly what we had before in the indeterminate case of the linear complex.

When

$$\Omega_i(aa) \equiv 0,$$

the hypercomplex consists of circles in involution with a given circle and the axial cross has disappeared.

Two proper crosses will have a complex invariant which we may write

$$H = \frac{\Sigma\left(a_{ij} + \frac{A_{ij}}{\sqrt{J}}\,\delta\right)\left(b_{ij} + \frac{B_{ij}}{\sqrt{J'}}\,\delta\right)}{\sqrt{\Sigma\left(a_{ij} + \frac{A_{ij}}{\sqrt{J}}\,\delta\right)^2}\,\sqrt{\Sigma\left(b_{ij} + \frac{B_{ij}}{\sqrt{J'}}\,\delta\right)^2}}.$$

We find the interpretation thereof in terms of the circles of the crosses as follows. We may replace our hypercomplexes by such other hypercomplexes

with the same axial crosses that these first invariants I and I' are zero.

$$\alpha_{ij} = ia_{ij} + \frac{A_{ij}}{\sqrt{J}}, \qquad \alpha'_{ij} = ia_{ij} - \frac{A_{ij}}{\sqrt{J}},$$

$$\beta_{ij} = ib_{ij} + \frac{B_{ij}}{\sqrt{J'}}, \qquad \beta'_{ij} = ib_{ij} - \frac{B_{ij}}{\sqrt{J'}},$$

$$\Sigma\alpha_{ij}^2 = 2i\sqrt{J}, \qquad \Sigma\alpha_{ij}'^2 = -2i\sqrt{J},$$
$$\Sigma\beta_{ij}^2 = 2i\sqrt{J'}, \qquad \Sigma\beta_{ij}'^2 = -2i\sqrt{J'},$$

$$I(\alpha\beta) - I(\alpha'\beta') = \frac{\Sigma\left[a_{ij}\frac{B_{ij}}{\sqrt{J'}} + b_{ij}\frac{A_{ij}}{\sqrt{J}}\right]}{\sqrt[4]{J}\sqrt[4]{J'}},$$

$$I(\alpha\beta') - I(\alpha'\beta) = \frac{\Sigma\left[a_{ij}b_{ij} + \frac{A_{ij}}{\sqrt{J}}\frac{B_{ij}}{\sqrt{J'}}\right]}{\sqrt[4]{J}\sqrt[4]{J'}},$$

$$H = \tfrac{1}{2}\{[I(\alpha\beta) - I(\alpha'\beta')] + [I(\alpha\beta') - I(\alpha'\beta)]\delta\}.$$

The strict truth is that we may so choose between the two circles of each cross that this last equation shall be fulfilled.

We pass now to a discussion of the axial crosses of the simplest one and two parameter families of linear hypercomplexes. The following theorem becomes evident when we remember that the central sphere of a linear hypercomplex is $x_4 = 0$ when, and only when $a_{i4} \equiv 0$.

THEOREM 26: *If two general linear hypercomplexes have the same central sphere, then this is the central sphere of every linear hypercomplex of the pencil linearly dependent on them. The hypercomplexes of the pencil will either all have the same axial crosses, or else these crosses will generate a cyclide or pseudo-cylindroid.*

The general case of a pencil of linear hypercomplexes is incredibly harder to handle. Let us write

(36)
$$c_{ij} = ya_{ij} + b_{ij},$$
$$\Omega_i(aa) \not\equiv \rho\Omega_i(bb),$$
$$\Omega_i(cc) = y^2\Omega_i(aa) + 2y\Omega_i(ab) + \Omega_i(bb).$$

The central spheres will all pass through two points while their centers (in euclidean space) will generate a conic. The spheres will thus generate a two-horn cyclide. We have the further equations

$$\Sigma c_{ij}^2 = y^2 I + 2y\Sigma a_{ij}p_{ij} + I',$$

$$\Sigma\Omega_i^2(cc) = y^4 J + 4y^3\Sigma\Omega_i(aa)\Omega_i(ab) + 2y^2[2\Sigma\Omega_i^2(ab) + \Sigma\Omega_i(aa)\Omega_i(bb)] + 4y\Sigma\Omega_i(ab)Q_i(bb) + J'.$$

If (γ) be an axial circle

$$\gamma_{ij} = xc_{ij} + C_{ij},$$

x must be the solution of an equation like (35), namely

$$(37)\quad \begin{aligned} &x^2 + 2x[y^2 I + 2y\Sigma a_{ij} b_{ij} + I'] + y^4 J + 4y^3 \Sigma\Omega_i (aa)\,\Omega_i (ab) \\ &+ 2y^2 [2\Sigma\Omega_i (ab)^2 + \Sigma\Omega_i (aa)\,\Omega_i (ab)] + 4y\Sigma\Omega_i (ab)\,\Omega_i (bb) + J' = 0. \end{aligned}$$

Each axial circle of our series will correspond to a pair of values of x and y satisfying these equations, each such pair of values will give one of our circles. The two values of x corresponding to the same value of y will give the two circles of a cross. If

$$(\Sigma a_{ij} b_{ij})^2 \neq \Sigma a_{ij}^2 \Sigma b_{ij}^2,$$

which we may assume to be the case, we may take

$$I = I' = 0,$$

for I will vanish for two hypercomplexes of the pencil. Our equation then takes the form

$$(38)\quad (x + ly)^2 = f^{(4)} (y).$$

THEOREM 27: *If two general linear hypercomplexes do not have the same central sphere the axial circles of a pencil of such hypercomplexes linearly dependent on them will generate, in general, an elliptic series of circles.*

If γ_{ij} belong to an arbitrary linear hypercomplex

$$\Sigma s_{ij} \gamma_{ij} = 0$$

we have an equation of the type

$$(ay + b)\, x + \phi^{(3)} (y) = 0.$$

Eliminating x we get a sextic in y; each y will correspond to but one x.

THEOREM 28: *Six circles of the series will belong to an arbitrary linear hypercomplex.*

Let us find the order of the surface swept out by the circles. If (z) be a point on one of our circles

$$\sum_n \gamma_{in} z_n = \sum_n \gamma_{jn} z_n = 0.$$

These will yield two equations

$$(ay + b)\, x + \varphi^3 (y) = 0, \qquad (a' y + b')\, x + \varphi'^{(3)} (y) = 0.$$

The elimination of x and y between these two and (38) is equivalent to eliminating $x\ y$ from

$$(ay + b)\, x + \varphi^3 (y) = (a' y + b')\, x + \varphi'^{(3)} (y) = x^2 - \psi^{(4)} (y) = 0.$$

These yield the equations

$$(ay+b)^2\psi^{(4)}(y) = [\phi^{(3)}]^2, \qquad (ay+b)\phi'^{(3)} - (a'y+b')\phi^{(3)} = 0.$$

If these latter be satisfied, so are the original three, or else

$$(ay+b) = \varphi^{(3)} = 0.$$

The resultant of a quartic and a sextic in (y) whose coefficients are quadratic in (z) will be an equation of the twentieth degree in (z). This will contain the resultant of a linear and a cubic equation, which is of the fourth degree in (z), as a quadratic factor. We must therefore split off from our expression of the twentieth degree one of the eighth degree.

THEOREM 29: *If two linear hypercomplexes have different central spheres, the axial circles of the pencil of linear hypercomplexes linearly dependent on them will generate a surface of the twenty-fourth order.*

We pass now to the general net of linear hypercomplexes, i. e., those linearly dependent on three whereof no two have the same central sphere. Here we have

$$d_{ij} = ya_{ij} + zb_{ij} + c_{ij},$$

$$\delta_{ij} = xd_{ij} + D_{ij},$$

$$x^2 + 2x[y^2\Sigma a_{ij}^2 + z^2\Sigma b_{ij}^2 + 2yz\Sigma a_{ij}b_{ij} + 2y\Sigma a_{ij}c_{ij} + 2z\Sigma b_{ij}c_{ij} + \Sigma c_{ij}^2] + \phi^{(4)}(yz) = 0.$$

We may, in general, assume that

$$\Sigma a_{ij}^2 = \Sigma b_{ij}^2 = \Sigma a_{ij}b_{ij} = 0.$$

Our equation takes the reduced form, in homogeneous coördinates

$$(39) \qquad (x+ly+mz)^2t^2 + f^{(4)}(yzt) = 0.$$

This may be linearly transferred to

$$x'^2t'^2 - (ay'^2 + by't' + ct'^2)^2 + z'f^3(y'z't') = 0.$$

The circles of our axial congruence are in one-to-one correspondence with the points of a quartic surface which has no singularity except the extremity of one axis. The really significant feature of this surface is that it is rational and can be carried over into a cubic surface by a quadratic Cremona transformation.*

$$\rho X = x't' + ay'^2 + by't' + ct'^2, \qquad \sigma x' = XZ - (aY^2 + bYT + cT^2),$$
$$\rho Y = y'z', \qquad \sigma y' = YT,$$
$$\rho Z = z'^2, \qquad \sigma z' = ZT,$$
$$\rho T = z't', \qquad \sigma t' = T^2.$$

* Cf. CREMONA: *Sopra un certo superficie di quart'ordine*, Collectanea Matematica, Naples, 1881.

Here it it must be remembered that $y'\,z'\,t'$ are linear and homogeneous in our original yzt. The cubic surface may be expressed parametrically by putting $XYZT$ as functions quadratic in each of two independent variables $u\,v$. Hence for our quartic

$$x = f^4(uv), \qquad y = \phi f_1^{(2)}(uv), \qquad z = \phi f_2^{(2)}(uv), \qquad f = \phi f_3^{(2)}(uv).$$

Now d_{ij} is linear and homogeneous in yzt, and the common factor ϕ may be struck therefrom, we thus get for the circles of our congruence the parametric expressions

$$\delta_{ij} = f^4(uv) f_{ij}^{(2)}(uv) + \phi_{ij}^{(6)}(uv).$$

It must be remembered that here the superscript indicates the degree in u and v separately, not the total degree.

THEOREM 32: *If no two linear hypercomplexes of a net have the same central sphere the coördinates of their axial circles may be expressed as rational functions of two independent variables each entering to the sixth order; seventy-two of these circles will belong to two arbitrary linear hypercomplexes.*

We find the focal surface of the congruence as follows. A sphere (z) will be orthogonal to a circle (δ) if

$$\sum_n \delta_{in} z = \sum \delta_{jn} z_n = 0, \qquad \delta_{ij} \neq 0.$$

We may for an arbitrary (z) surely choose i and j so that these three conditions are satisfied. The equation which we desire is obtained by equating to zero the tact-invariant of these looked upon as two curves in the uv plane. The order of this in the coefficients of one of the curves is the number of free intersections of this curve with the Jacobian of the two given curves and a third of the same structure.* Each of our curves is of the twelfth order with a sextuple point at the end of each axis. The Jacobian is of the thirty-third order, but contains u and v only to the seventeenth order; it has therefore a point of the sixteenth order at the extremity of each axis. The Jacobian intersects each curve three hundred ninety-six times, but one hundred ninety-two intersections are at the ends of the axes; there remain two hundred four free intersections; this is therefore the order of the tact invariant in the coefficients of each curve, or one half of its order in (z).

THEOREM 33: *The axial circles of a general net of linear hypercomplexes envelop a surface of the order eight hundred sixteen.*

There is more satisfaction in finding the order of such a surface than in seeking its equation.

The general cubic surface contains twenty-seven straight lines, and these transform into parabolas of the quartic surface. The plane of each of these parabolas will meet the quartic in a second parabola with a parallel axis. There was one parabola common to all the homaloidal paraboloids of the

* Cf. SALMON, *Higher Plane Curves*, third ed., p. 81.

Cremona transformation, and another in its plane. Beyond these fifty-six there are no parabolas on the quartic.* Each of these may be expressed in the form

$$x = \phi^{(2)}(s), \qquad y = (as+b)(a_1 s_1 + b_1), \qquad z = (as+b)(a_2 s + b_2),$$
$$t = (as+b)(a_3 s + b_3).$$

We see that d_{ij} is linear, and δ_{ij} cubic in s.

THEOREM 34: *The congruence contains fifty-six surfaces of the twelfth order.*

§ 4. *An Involutory Circle Transformation.*

If two non-cospherical circles (p) and (q) be given, there will be just one circle orthogonal to their common orthogonal sphere, and in bi-involution with (p). The coördinates of this sphere are $\Omega_i(pq)$; we wish for the circle orthogonal to this and to the two null spheres (x) and (y) whose vertices are the foci of (p). The circle required will thus, by (23), have the Plücker coördinates

$$P_{ij} = \begin{vmatrix} x_k & x_l & x_m \\ y_k & y_l & y_m \\ \Omega_k & \Omega_l & \Omega_m \end{vmatrix},$$

$$P_{ij} = p_{lm}\Omega_k(pq) + p_{mk}\Omega_l(pq) + p_{kl}\Omega_m(pq). \tag{40}$$

If a fundamental circle (q) be given, then each circle (p) not cospherical with it will correspond in an involutory way to a circle (P), the relation between the two being that they are in bi-involution, and their common orthogonal sphere is orthogonal to (q). Another way of stating this is to say that (P) is the circle through the foci of (p) orthogonal to the sphere through those foci and the foci of the fundamental circle. The relation of (p) to (P) is involutory. The most striking feature about this transformation, however, is that it does not carry cospherical circles into cospherical circles.† We find in fact

$$\Omega_m(PP') = \begin{vmatrix} 0 & x_i & x_j & x_k & x_l & x_m \\ 0 & y_i & y_j & y_k & y_l & y_m \\ x'_m & x'_i & x'_j & x'_k & x'_l & 0 \\ y'_m & y'_i & y'_j & y'_k & y'_l & 0 \\ 0 & \Omega_i(pq) & \Omega_j(pq) & \Omega_k(pq) & \Omega_l(pq) & \Omega_m(pq) \\ \Omega_m(p'q) & \Omega_i(p'q) & \Omega_j(p'q) & \Omega_k(p'q) & \Omega_l(p'q) & 0 \end{vmatrix}.$$

* CREMONA, loc. cit., p. 418.

† Previous writers do not seem to have noticed circle transformations which were not also sphere transformations. It is the more remarkable that the first of these to be studied should be involutory.

Assuming that (p) and (p') are cospherical so that $\Omega_i(pp') \equiv 0$, we have

$$\Omega_m(PP') = \Sigma[\Omega_l(pq)\,\Omega_m(p'q) - \Omega_m(pq)\,\Omega_l(p'q)]$$
$$[p_{jk}\,p'_{mi} + p_{ki}\,p'_{mj} + p_{ij}\,p'_{mk}]$$
$$+ \Sigma[\Omega_j(pq)\,\Omega_k(p'q) - \Omega_k(pq)\,\Omega_j(p'q)][p'_{mi}\,p_{lm} - p_{mi}\,p'_{lm}].$$

This will vanish for all values of m and for all pairs of cospherical circles when

$$\Omega_i(pq) \equiv \Omega_i(p'q).$$

THEOREM 35: *The transformation which carries each circle not cospherical with a fundamental circle into the circle in bi-involution with it, and orthogonal to the sphere which is orthogonal to it and to the fundamental circle, is single valued and involutory, but will in general carry cospherical circles into cospherical circles when, and only when, the common sphere is orthogonal to the fundamental circle.*

It will be found also that the properties of involution and bi-involution are not invariant under the transformation, neither is it usually a contact transformation.

Suppose that (p) traces a pencil of circles including one that is cospherical with (q_j); we shall have

$$P_{ij} = \lambda A_{ij} + \mu B_{ij}.$$

When no member of the pencil is cospherical with (q) we have

$$P_{ij} = \lambda^2 R_{ij} + 2\lambda\mu A_{ij} + \mu^2 S_{ij}.$$

All of these circles are orthogonal to the sphere generated by the pencil. There are but three conditions imposed upon a circle orthogonal to this sphere to require it to be cospherical with all of the circles of this system.

THEOREM 36: *A pencil of circles containing one member cospherical with the fundamental circle is transformed into another such pencil; a pencil of circles containing no such member is transformed into one generation of a cyclide anallagmatic in the sphere generated by the given pencil.*

It would be pleasant if the circles of the pencil cospherical and orthogonal to the first transformed into the conjugate generation of the same cyclide; such is not, however, the case. We find the transforms of the circles of a sphere which does not contain (q) as follows. Let us take as the sphere $x_4 = 0$, which we may assume has no special relation to (q); (p) will have only those coördinates which have a subscript 4, P will lack all coördinates with this subscript, so that (P) is orthogonal to that sphere. We find also, in the notation of section 2, that $(P/q) = 0$.

THEOREM 37: *The circles of a sphere not containing the fundamental circle are transformed into the circles orthogonal to that same sphere and cospherical with the circle which meets it orthogonally in the same points as the fundamental circle.*

This theorem will hold by continuity even when the sphere is null. If the circles on the given sphere be orthogonal to a fixed circle one more linear condition is imposed on (P). If two conditions of this type be imposed upon (P) we have, by (36), one generation of a cyclide, hence

THEOREM 38: *The circles of a sphere which does not contain the fundamental circle which cut orthogonally a fixed circle of that sphere will be transformed into circles cospherical with two fixed circles orthogonal to that sphere.*

Suppose that we have a system of circles through two points. Their foci trace the circle whose foci are these two points, and any two points of this circle are foci of a circle through the original two points.

THEOREM 39: *The circles through two fixed points, not on the fundamental circle, are transformed into the circles through each pair of points of the circle whose foci are the fixed points, orthogonal in each case to the sphere connecting the two required points with the foci of the fundamental circle.*

THEOREM 40: *The circles orthogonal to a sphere which does not contain the fundamental circle are transformed into the circles through pairs of points of this sphere, orthogonal in each case to the sphere connecting the pairs of points with the foci of the fundamental circle.*

Let us find the transforms of all circles which are cospherical with a given circle (r). Since (p) and (r) are cospherical, their foci are concyclic; (p) transforms into the circle through its foci, orthogonal to the sphere containing them and the foci of (q). If we take an arbitrary sphere through the foci of (q), the points thereon concyclic with the foci of (r) are corresponding points in the inversion whose fundamental sphere is the sphere through (r) orthogonal to the sphere in question.

THEOREM 41: *The circles cospherical with a given circle not cospherical nor in bi-involution with the fundamental circle are the circles orthogonal to each sphere through the foci of the fundamental circle at pairs of points which are mutually inverse in the sphere through the given circle and orthogonal to the sphere in question.*

When (q) and (r) are cospherical, every circle through the foci of (r) lies on a sphere through the foci of (q).

THEOREM 42: *The circles cospherical with a circle which is cospherical with the fundamental circle transform into the circles in involution with the given circle and with the circle containing the foci of the given and the fundamental circle.*

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.,
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PROJECTIVE DIFFERENTIAL GEOMETRY OF DEVELOPABLE SURFACES*

BY

WILLIAM WELLS DENTON

§ 1. *The simultaneous solutions of an involutory system of two linear homogeneous partial differential equations of the second order, with two independent variables, and a similar equation of the third order.*

Let there be given a system of partial differential equations of the form

$$\begin{aligned} &Ay_{uu} + By_{uv} + Cy_{vv} + Dy_u + Ey_v + Fy = 0, \\ &A'y_{uu} + B'y_{uv} + C'y_{vv} + D'y_u + E'y_v + F'y = 0, \end{aligned} \tag{1}$$

where

$$y_u = \frac{\partial y}{\partial u}, \qquad y_v = \frac{\partial y}{\partial v}, \qquad y_{uv} = \frac{\partial^2 y}{\partial u \partial v}, \qquad \text{etc.},$$

and where $A, B, \cdots F'$ are analytic functions of u and v, such that not all of the second order determinants of the matrix

$$\left\| \begin{matrix} A & B & C \\ A' & B' & C' \end{matrix} \right\| \tag{2}$$

vanish.

Differentiation of these equations with respect to u and v gives

$$\begin{aligned} &Ay_{uuu} + By_{uuv} + Cy_{uvv} + * + (A_u + D)y_{uu} + (B_u + E)y_{uv} \\ &\qquad + C_u y_{vv} + (D_u + F)y_u + E_u y_v + F_u y = 0, \\ &* + Ay_{uuv} + By_{uvv} + Cy_{vvv} + A_v y_{uu} + (B_v + D)y_{uv} \\ &\qquad + (C_v + E)y_{vv} + D_v y_u + (E_v + F)y_v + F_v y = 0, \\ &A'y_{uuu} + B'y_{uuv} + C'y_{uvv} + * + (A'_u + D')y_{uu} \\ &\qquad + (B'_u + E')y_{uv} + C'_u y_{vv} + (D'_u + F')y_u + E'_u y_v + F'_u y = 0, \\ &* + A'y_{uuv} + B'y_{uvv} + C'y_{vvv} + A'y_{uu} + (B'_v + D')y_{uv} \\ &\qquad + (C'_v + E')y_{vv} + D'_v y_u + (E'_v + F')y_v + F'_v y = 0, \end{aligned} \tag{3}$$

so that it is possible to express the third, and indeed all higher derivatives of

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y in the form

$$\alpha y_{uu} + \beta y_{uv} + \gamma y_{vv} + \delta y_u + \epsilon y_v + \zeta y, \tag{4}$$

provided the determinant of the coefficients,

$$\begin{vmatrix} A & B & C & 0 \\ 0 & A & B & C \\ A' & B' & C' & 0 \\ 0 & A' & B' & C' \end{vmatrix} = K,$$

is not identically equal to zero.

The theory of the invariants and covariants of system (1), in the case that K does not vanish identically, forms a basis for the projective differential geometry of non-developable surfaces. That theory has been developed by Professor WILCZYNSKI.* The object of this paper is to give a similar theory for developable surfaces.† The writer takes this opportunity to thank Professor WILCZYNSKI for personally directing this work.

We shall consider a system of form (1) for which K is identically equal to zero. Since not all of the second order determinants of the matrix (2) vanish, all of the third order minors of K cannot be identically equal to zero. Therefore, if, in addition to equations (1), any linear homogeneous partial differential equation of the third order,

$$Gy_{uuu} + Hy_{uuv} + Iy_{uvv} + Jy_{vvvv} + A'' y_{uu} + B'' y_{uv} + C''' y_{vv} + D'' y_u + E'' y_v + F'' y = 0, \tag{5}$$

independent of equations (3) be given, it will again be possible to express all the derivatives of y in the form (4).

The new system, composed of a system of equations of form (1), for which K vanishes identically, and of equation (5), and for which certain integrability conditions, to be discussed presently, are satisfied, will be called system (A). The theory of the invariants and covariants of this system forms a basis for the projective differential geometry of developable surfaces, as will be shown.

Let y be a solution of system (A), analytic in the vicinity of (u_0, v_0), so that it may be written

$$y = y^0 + y^0_u (u - u_0) + y^0_v (v - v_0) + \tfrac{1}{2}[\, y^0_{uu} (u - u_0)^2 + 2y^0_{uv} (u - u_0)(v - v_0) + y_{vv} (v - v^0)^2] + \cdots, \tag{6}$$

* E. J. WILCZYNSKI, *Projective Differential Geometry of Curved Surfaces.* See especially *Fourth memoir*, these Transactions, vol. 10 (1909), p. 76. See also preceding memoirs, vol. 8 (1907), p. 233; vol. 9 (1908), p. 79; p. 293.

† Unless otherwise stated, the term developable surfaces will be understood to include cones.

where y^0, y^0_u, etc., are the values of y, y_u, $\cdots$, for $u = u_0$, $v = v_0$. Since all derivatives of y can be expressed uniquely in the form (4), the series (6) may be put in the form

$$y = y^0 y' + y^0_u y'' + y^0_v y^{(3)} + y^0_{uu} y^{(4)} + y^0_{uv} y^{(5)} + y^0_{vv} y^{(6)}, \tag{7}$$

where y', y'', $\cdots y^{(6)}$ are analytic functions of $u - u_0$, $v - v_0$. The number of linearly independent analytic solutions of (A) is equal to the number of arbitrary constants in the general solution (7). On account of equations (1) not all of the six constants, y_0, y^0_u, $\cdots y^0_{vv}$, are arbitrary. The following system of equations is always equivalent to system (1),

$$\begin{aligned} &* + (AB)\, y_{uv} + (AC)\, y_{vv} + (AD)\, y_u + (AE)\, y_v + (AF)\, y = 0, \\ &(BA)\, y_{uu} + * + (BC)\, y_{vv} + (BD)\, y_u + (BE)\, y_v + (BF)\, y = 0, \\ &(CA)\, y_{uu} + (CB)\, y_{uv} + * + (CD)\, y_u + (CE)\, y_v + (CF)\, y = 0, \end{aligned} \tag{8}$$

where $(AB) = AB' - A'B$, etc. The following cases may be distinguished.

I. If the determinants (AB), (BC), (CA) are all different from zero, each of the second derivatives of y may be expressed linearly in terms of any one of them and y, y_u, y_v; so that every expression of form (4) may be reduced to any one of the following forms

$$\alpha_1 y_{uu} + \beta_1 y_u + \gamma_1 y_v + \delta_1 y, \tag{9}$$

$$\alpha_2 y_{uv} + \beta_2 y_u + \gamma_2 y_v + \delta_2 y, \tag{10}$$

$$\alpha_3 y_{vv} + \beta_3 y_u + \gamma_3 y_v + \delta_3 y. \tag{11}$$

Since $K = (CA)^2 - (AB)(BC)$ is identically equal to zero, if one of the three determinants (AB), (BC), (CA) vanishes, at least one of the others must also be equal to zero. We therefore consider the following additional possibilities.

II. If (AB) and (BC) were identically equal to zero, (CA) would also be identically equal to zero, and the above equations (8) would reduce to the first order or disappear. Both possibilities have been excluded by the assumption that the second order determinants of the matrix (2) are not all equal to zero.

III. If (BC) and (CA) are identically equal to zero, while (AB) is different from zero, every expression of form (4) may be reduced to form (11).

IV. If (CA) and (AB) are identically equal to zero, while (BC) is different from zero, every expression of form (4) may be reduced to form (9).

It is to be noticed that forms (9) and (11) are identical, except for an interchange of the variables u and v, so that cases III and IV may be regarded as identical except for the notation.

Multiplying equations (3), in order, by the minors of the first, second, third and fourth elements in the first column of K, and adding, we find

$$A_1 y_{uu} + B_1 y_{uv} + C_1 y_{vv} + D_1 y_u + E_1 y_v + F_1 y = 0, \tag{12}$$

where

$$A_1 = \begin{vmatrix} A_u + D & B & C & 0 \\ Av & A & B & C \\ A_u + D' & B' & C' & 0 \\ A_v & A' & B' & C' \end{vmatrix}, \text{ etc.}$$

Let the left member of equation (12) be reduced by means of equations (8) to one of the above forms, (9), (10), or (11), so that (12) takes the form

$$\alpha w + \beta y_u + \gamma y_v + \delta y = 0, \tag{13}$$

where w stands for y_{uu}, y_{uv}, or y_{vv}. If all four of the coefficients of the new equation are not identically equal to zero, it is possible, by means of this equation, to reduce any expression of the chosen form to one involving not more than three terms. In this case system (A) cannot have more than three linearly independent analytic solutions. If all of the coefficients of equation (13) are identically equal to zero, equations (1) are said to be in *involution*. In this case there is a possibility that the four quantities y, y_u, y_v, w remain arbitrary, so that system (A) has four and only four linearly independent solutions.

It must be noticed, however, that some of the derivatives of y may be computed in more than one way; for example

$$y_{uvvv} = \frac{\partial y_{uvv}}{\partial v} = \frac{\partial y_{vvv}}{\partial u}.$$

In order, therefore, that the expressions of form (4) for the derivatives may be consistent, certain so-called *integrability conditions* must be satisfied. They are of the form (13). Let there be n independent conditions in the set, from which all others may be derived,

$$\alpha_k w + \beta_k y_u + \gamma_k y_v + \delta_k y = 0 \qquad (k = 1, 2, \cdots, n). \tag{14}$$

The number of linearly independent quantities, w, y_u, y_v, y, is reduced by at least one for each non-identically satisfied equation of the set (14). But if equations (14) are all identically satisfied, that is, if

$$\alpha_k = \beta_k = \gamma_k = \delta_k = 0 \qquad (k = 1, 2, \cdots, n),$$

all four quantities, w, y_u, y_v, y, may remain arbitrary, so that system (A) has four linearly independent solutions.

The following theorem may now be stated. *Given a system of partial differential equations of form* (A), *consisting of a system of form* (1), *for which* K *is identically equal to zero, and an equation of form* (5), *not derivable from equations* (1), *all of whose coefficients are analytic functions.* (α) *If the determinants* (AB), (BC), (CA) *are not all identically equal to zero,* (β) *if further the equations* (1) *are in involution,* (γ) *if finally the integrability conditions are identically satisfied, system* (A) *has four and only four linearly independent analytic solutions. If any one of the above conditions* (α), (β), (γ), *is not satisfied, the number of linearly independent solutions of system* (A) *is less than, or exceeds, four.*

To complete the proof of this theorem it suffices to apply well-known existence theorems.

§ 2. *Geometric interpretation. The integrating developable surface.*

If the conditions (α), (β), (γ) are satisfied, system (A) has precisely four linearly independent analytic solutions. Let these be

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4), \tag{15}$$

Interpret $y^{(k)}$ as the homogeneous coördinates of a point y in space. As u and v assume all of their values, y describes a surface S, an integral surface of the system. This surface cannot degenerate into a curve. For if S degenerated in that way, the ratios of $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$ would be functions of a single variable $t = \tau(u, v)$, and $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$, themselves would be solutions of some linear partial differential equations of the first order,* and this is contrary to the assumption that the system has four linearly independent solutions.

The surface S is not unique. The most general system of linearly independent analytic solutions of (A) is

$$\eta_i = \sum_{k=1}^{4} c_{ik} y_k \qquad (i = 1, 2, 3, 4),$$

where the determinant of the constant coefficients c_{ik} does not vanish. The most general integral surface of (A) is, therefore, a projective transformation of any particular one.

The integral surface is, moreover, a developable. This follows from the fact that equations (1) are in involution. Conversely, it can be shown that every developable is an integral surface of an involutory system of form (1). A proof of these statements has been given by Professor Wilczynski.†

Therefore, *the theory of a system of partial differential equations of form* (A), *for which the conditions* (α), (β), (γ) *are all satisfied, is identical with the projective differential geometry of developable surfaces in three-dimensional space.*

* Ibid., vol. 8, p. 237.

† Loc. cit., pp. 238–240.

§ 3. *First intermediate form. The surface referrred to its generators.*

For present purposes, it will be convenient to take system (A) in the following form

$$y_{uu} = ay_{vv} + by_u + cy_v + dy, \tag{16}$$

$$y_{uv} = a' y_{vv} + b' y_u + c' y_v + d' y, \tag{17}$$

$$y_{vvv} = a'' y_{vv} + b'' y_u + c'' y_v + d'' y, \tag{18}$$

which will hereafter be called system (B). It was shown in § 1 that system (A) can always be reduced to the form (B), except that if (CA) and (AB) are identically equal to zero, the variables u and v must be interchanged. The theory, therefore, can suffer no essential loss of generality in consequence of this choice of the form (B).

For system (B), the value of K is $a - a'^2$, and this is, of course, assumed to be identically equal to zero.

Differentiation of equations (16) and (17), with respect to u and v, gives

$$y_{uuu} - ay_{uvv} = \alpha_1 y_{vv} + \beta_1 y_u + \gamma_1 y_v + \delta_1 y, \tag{19}$$

$$y_{uuv} - a' y_{uvv} = \alpha_2 y_{vv} + \beta_2 y_u + \gamma_2 y_v + \delta_2 y, \tag{20}$$

$$y_{uuv} - ay_{vvv} = \alpha_3 y_{vv} + \beta_3 y_u + \gamma_3 y_v + \delta_3 y, \tag{21}$$

$$y_{uvv} - a' y_{vvv} = \alpha_4 y_{vv} + \beta_4 y_u + \gamma_4 y_v + \delta_4 y, \tag{22}$$

where $\alpha_1 = a_u + ab + ca'$, etc.

Multiplying equation (20) by a', (21) by $-a'$, (22) by a, and adding, we find

$$\begin{aligned}(a - a'^2)\, y_{uvv} = {} & (a'\alpha_2 - a'\alpha_3 + a\alpha_4)\, y_{vv} + (a'\beta_2 - a'\beta_3 + a\beta_4)\, y_u \\ & + (a'\gamma_2 - a'\gamma_3 + a\gamma_4)\, y_v + (a'\delta_2 - a'\delta_3 + a\delta_4)\, y.\end{aligned} \tag{23}$$

Since equations (16) and (17) are in involution, all of the coefficients in equation (23) must be identically equal to zero.

It will not be necessary to develop the integrability conditions explicitly at the present moment. It is assumed that they are satisfied identically, so that system (B) has four and only four linearly independent analytic solutions.

If a surface is represented by equations of the form (15), a transformation of the form

$$\bar{u} = \phi(u, v), \qquad \bar{v} = \psi(u, v), \tag{24}$$

will modify at most the parametric curves on the surface but not the surface itself.

Let the transformation (24) be applied to system (B). If the resulting system be solved for $\bar{y}_{uu}$ and $\bar{y}_{uv}$, the new coefficients corresponding to a

and a' are found to be

$$(25)\quad \bar{a} = \frac{\psi_u^2 - 2a'\psi_u\psi_v + a\psi_v^2}{\phi_u^2 - 2a'\phi_u\phi_v + a\phi_v^2}, \qquad \bar{a}' = \frac{\phi_u\psi_u + a\phi_v\psi_v - a'\phi_u\psi_v - a'\phi_v\psi_u}{\phi_u^2 - 2a'\phi_u\phi_v + a\phi_v^2},$$

which give $\bar{a} - \bar{a}'^2 = 0$, as they should. The other coefficients of the new equations of the second order are fractions with the common denominator

$$\Delta = (\phi_u\psi_v - \phi_v\psi_u)(\phi_u^2 - 2a'\phi_u\phi_v + a\phi_v^2);$$

for example,

$$(26)\quad \begin{aligned}\Delta\bar{c} = {} & (-\psi_{uu} + a\psi_{vv} + a'b\psi_v + c\psi_v)(\phi_u\psi_v + a'\psi_v\phi_v - 2a'\phi_v\psi_v) \\ & - 2(-a\phi_v\psi_v + a'\phi_u\psi_v)(-\psi_{uv} + a'\psi_{vv} + a'b'\psi_v + c'\psi_v).\end{aligned}$$

The coefficients of the new equation of the third order, obtained by solving the transforms of equations (18), (19), (21), (22) for $\bar{y}_{vvv}$ are fractions with the common denominator $(\phi_u\psi_v - \phi_v\psi_u)^6$.

If ψ be chosen so as to satisfy the equation

$$(27)\quad \psi_u\psi_v^{-1} = a',$$

the numerators of $\bar{a}$ and $\bar{a}'$ will both vanish. The denominators,

$$\phi_u^2 - 2a'\phi_u\phi_v + a\phi_v^2 = (\phi_u - a'\phi_v)^2,$$

however, cannot vanish; for, if this expression were equal to zero, it would be necessary to have

$$\phi_u\phi_v^{-1} = a' = \psi_u\psi_v^{-1}, \quad \text{or} \quad \phi_u\psi_v - \phi_v\psi_u = 0,$$

contrary to the assumption that ϕ and ψ are independent functions of u and v. If, therefore, the transformation be subjected to the single condition (27), the new equations of the second order will be free from the terms in $\bar{y}_{vv}$.

If $\bar{a} = \bar{a}' = 0$, the original system has the desired form without transformation. Suppose this to be the case, or imagine that a first transformation of this character has been made. Then the system to be considered has the form

$$(28)\quad y_{uu} = by_u + cy_v + dy,$$

$$(29)\quad y_{uv} = b'y_u + c'y_v + d'y,$$

$$(30)\quad y_{vvv} = a''y_{vv} + b''y_u + c''y_v + d''y.$$

This will be called system (C), or the *first intermediate form* of system (B).

Equation (23) shows that the two equations of the second order, which appear in this system, are always in involution. It is not necessary, therefore, from now on, to pay attention to the involutorial conditions.

The integrability conditions for this new system require that c be equal to zero identically. This is seen at once, by differentiating equation (28) with respect to v, (29) with respect to u, and equating the coefficients of y_{vv} in the resulting expressions for y_{uvv}.

Since c is zero, equation (28) shows that the curves $v = \text{const.}$, upon the integral surfaces of system (C), are straight lines. *The reduction of a system of form (A) to form (C) is, therefore, equivalent to the determination of the straight line generators of its developable integral surfaces.*

§ 4. *Second intermediate form. The surface referred to its plane sections.*

Equations (25) and (26) show that the form of system (C) is not changed by any transformation of the form

$$\bar{u} = \phi(u, v), \qquad \bar{v} = \psi(v), \tag{31}$$

where ϕ and ψ are arbitrary functions of the arguments indicated. Making this transformation, and solving the resulting equations for $\bar{y}_{vvv}$, we find for the new coefficient corresponding to b'' in the original system,

$$\overline{b''} = \phi_u \psi_v^{-3} (b'' + \chi), \tag{32}$$

where

$$\begin{aligned}\chi = {} & \theta (c'' + 2a'' b' - 3b_v - 3b'^2) \\ & + \theta^2 (2a'' c' - a'' b - 9b' c' + 3b_v + 3bb' - 3c_v' - 3d') \\ & + \theta^3 (3bc' - 6c'^2 - 3b^2 - 3b_u - 3d) \\ & + \theta^2 \eta (a'' + 6c' - 6b) + a'' \theta^2 \eta^2 - 12c' \theta^3 \eta \\ & + 9\theta^3 \eta^2 - \theta_{vv} + 2\theta\theta_{uv} + a'' \theta_{uv} + \theta_u \theta_v - \theta\theta_u^2 + a'' \theta_u^2 \\ & + \theta\theta_v (-6c' - 3b) - 3b' \theta_v + 3b' \theta\theta_u - 9b\theta^2 \theta_u + 6\theta\theta_v \eta \\ & + 10\theta^2 \theta_u \eta + 2a'' \theta\theta_u\eta + a'' \theta_v \eta + \theta^3 \eta_u + 5\theta^2 \eta_v + a'' \theta\eta_v,\end{aligned}$$

$$\theta = \phi_v \phi_u^{-1}, \qquad \eta = \phi_{uu} \phi_u^{-1}.$$

It is possible, therefore, to make $\overline{b''} = 0$, by solving a differential equation of the third order for the function ϕ, viz.,

$$b'' + \chi = 0. \tag{33}$$

Imagine this to have been done, or that b'' is already equal to zero. Then the system of equations under discussion has the following form,

$$y_{uu} = by_u + dy, \tag{34}$$

$$y_{uv} = b' y_u + c' y_v + d' y, \tag{35}$$

$$y_{vvv} = a'' y_{vv} + c'' y_v + d'' y, \tag{36}$$

which will be called form (D), or the *second intermediate form* of system (B).

Equation (36) shows that the curves $u = \text{const.}$, upon the integral surfaces of system (D), are plane curves. Conversely, if the curves $u = \text{const.}$ are plane curves, the four linearly independent analytic functions, $y^{(1)}$, $y^{(2)}$, $y^{(3)}$, $y^{(4)}$, must satisfy an equation of this form. Therefore, *the reduction of system (C) to a system of form (D), or the reduction of system (A) to a system which contains an equation of form* (36), *is equivalent to the determination of a one-parameter family of plane sections of the integral developable surfaces of the original system.*

Whenever the generators and a one parameter family of plane sections of a developable are given there may be constructed immediately a system of form (D) satisfied by the developable. If, however, a general system of form (A) be given, so that the finite equations of its generators and plane sections are unknown, their determination with reference to the known elements requires the integrations involved in reducing the given system of form (A) to the second intermediate form.

§ 5. *The integrability conditions for a system in the second intermediate form. The canonical form.*

It will be necessary to develop explicitly the integrability conditions for system (D). These are obtained by putting

$$\frac{\partial y_{uu}}{\partial v} = \frac{\partial y_{uv}}{\partial u}, \qquad \frac{\partial y_{vvv}}{\partial u} = \frac{\partial^2 y_{uv}}{\partial v^2}. \tag{37}$$

From equations (34), (35), (36), we find by differentiation,

$$\begin{aligned}
y_{uuv} &= (b'_v + bb')\, y_u + (d + bc')\, y_v + (d_v + bd')\, y, \\
y_{uvu} &= (b'_u + bb' + c'\, b' + d')\, y_u + (c'_u + c'^2)\, y_v + (b'\, d + c'\, d' + d'_u)\, y, \\
y_{vvvu} &= (a''_u + c'\, a'')\, y_{vv} + [a''\,(b'_v + b'^2) + b'\, c'' + d'']\, y_u \\
&\quad + [a''\,(c'_v + d' + b'\, c') + c''_u + c'\, c'']\, y_v \\
&\quad + [a''\,(d'_v + b'\, d') + c''\, d' + d''_u]\, y, \\
y_{uvvv} &= (2c'_v + d' + b'\, c' + a''\, c')\, y_{vv} + [(b'_v + b'^2)_v + b'\,(b'_v + b'^2)]\, y_u \\
&\quad + [c'\,(b'_v + b'^2) + (c'_v + d' + b'\, c')_v + d'_v + b'\, d' + c'\, c'']\, y_v \\
&\quad + [d'\,(b'_v + b'^2) + (d'_v + b'\, d')_v + c'\, d'']\, y.
\end{aligned}$$

Equations (37) ensure unique values for all third and fourth order derivatives. That the expressions for all higher order derivatives will also be unique then follows directly by differentiation. Since these equations must hold for all

values of y_{vv}, y_u, y_v, y, it is necessary to have,

$$
\begin{aligned}
&b_v = b'_u + b'\,c' + d', \\
&d + bc' = c'_u + c'^2, \\
&bd' + d_v = d'_u + c'\,d' + b'\,d, \\
&a''_u = 2c'_v + d' + b'\,c', \\
&a''\,(b'_v + b'^2) + b'\,c'' + d'' = (b'_v + b'^2)_v + b'\,(b'_v + b'^2), \\
&a''\,(b'\,c' + d' + c'_v) - (b'\,c' + d' + c'_v)_v - (b'\,d' + d'_v) + c''_u \\
&\qquad\qquad - c'\,(b'_v + b'^2) = 0, \\
&a''\,(b'\,d' + d'_v) + c''\,d' - c'\,d'' + d''_u - d'\,(b'_v + b'^2) - (b'\,d' + d'_v)_v = 0.
\end{aligned}
\tag{38}
$$

The first and fourth of these equations give

$$(a'' + b')_u = (2c' + b)_v. \tag{39}$$

It is, therefore, possible to find a function p, of u and v, satisfying the following conditions,

$$p_u = 2c' + b, \qquad p_v = a'' + b'. \tag{40}$$

Transform system (D) by putting

$$y = \lambda \bar{y}, \tag{41}$$

where λ is an arbitrary function of u and v. Since the coördinates are homogeneous, the integral surface of the system will remain unchanged under this transformation. The form of the system, moreover, remains unchanged, and the coefficients of the new equations are found to be

$$
\begin{aligned}
&\bar{b} = b - 2\lambda_u \lambda^{-1}, \\
&\bar{d} = d + b\lambda_u \lambda^{-1} - \lambda_{uu} \lambda^{-1}, \\
&\bar{b}' = b' - \lambda_v \lambda^{-1}, \\
&\bar{c}' = c' - \lambda_u \lambda^{-1}, \\
&\bar{d}' = d' + c'\,\lambda_v \lambda^{-1} - \lambda_{uv} \lambda^{-1} + b'\,\lambda_u \lambda^{-1}, \\
&\bar{a}'' = a'' - 3\lambda_v \lambda^{-1}, \\
&\bar{c}'' = c'' + 2a''\,\lambda_v \lambda^{-1} - 3\lambda_{vv} \lambda^{-1}, \\
&\bar{d}'' = d'' + a''\,\lambda_{vv} \lambda^{-1} + c''\,\lambda_v \lambda^{-1} - \lambda_{vvv} \lambda^{-1}.
\end{aligned}
\tag{42}
$$

The expressions corresponding to p_u and p_v for the new equations are

$$\bar{p}_u = 2\bar{c}' + \bar{b} = 2c' + b - 4\lambda_u \lambda^{-1}, \quad \bar{p}_v = a'' + \bar{b}' = a'' + b' - 4\lambda_v \lambda^{-1}, \tag{43}$$

so that they may be made to vanish by putting

$$y = \lambda \bar{y}, \qquad \lambda = e^{p/4}. \tag{44}$$

Let this restriction be placed upon the transformation (41) which has just been applied to system (D). Then the new coefficients, which will be denoted by German letters, are found, by specializing equations (42), to be

$$\begin{aligned}
&\mathfrak{b} = b - 2e, && \mathfrak{d}' = d' + c'e' - e_v - ee' + b'e,\\
&\mathfrak{d} = d + be - e_u - e^2, && \mathfrak{a}'' = a'' - 3e',\\
&\mathfrak{b}' = b' - e', && \mathfrak{c}'' = c'' + 2a''e' - 3e'_v - 3e'^2,\\
&\mathfrak{c}' = c' - e, && \mathfrak{d}'' = d'' + a''(e'_v + e'^2) + c''e' - e'_{vv}\\
& && \qquad - e'^3 - 3e'e'_v,
\end{aligned} \tag{45}$$

where

$$4e' = a'' + b', \qquad 4e = 2c' + b,$$

and, of course,

$$\mathfrak{b} = -2\mathfrak{c}', \qquad \mathfrak{a}'' = -\mathfrak{b}'.$$

The new system, which will be called (E), has the *canonical form*

$$y_{uu} = -2\mathfrak{c}'y_u + \mathfrak{d}y, \tag{46}$$

$$y_{uv} = \mathfrak{b}'y_u + \mathfrak{c}'y_v + \mathfrak{d}'y, \tag{47}$$

$$y_{vvv} = -\mathfrak{b}'y_{vv} + \mathfrak{c}''y_v + \mathfrak{d}''y. \tag{48}$$

Any system of form (A) can be reduced to such a canonical form. The integrability conditions for system (E) may be found from equations (38). They are as follows:

$$-2\mathfrak{c}'_v = \mathfrak{b}'_u + \mathfrak{b}'\mathfrak{c}' + \mathfrak{d}', \tag{49}$$

$$\mathfrak{d} = \mathfrak{c}'_u + 3\mathfrak{c}'^2, \tag{50}$$

$$\mathfrak{d}_v = \mathfrak{d}'_u + 3\mathfrak{c}'\mathfrak{d}' + \mathfrak{b}'\mathfrak{d}, \tag{51}$$

$$\mathfrak{b}'\mathfrak{c}'' + \mathfrak{d}'' = \mathfrak{b}'_{vv} + 4\mathfrak{b}'\mathfrak{b}'_v + 2\mathfrak{b}'^3, \tag{52}$$

$$\mathfrak{c}''_u = \mathfrak{c}'_{vv} + 2(\mathfrak{b}'\mathfrak{c}' + \mathfrak{d}')_v + 2(\mathfrak{b}'\mathfrak{c}' + \mathfrak{d}')\mathfrak{b}', \tag{53}$$

$$\mathfrak{d}''_u = \mathfrak{d}'_{vv} + 2(\mathfrak{b}'\mathfrak{d}')_v + 2\mathfrak{b}'^2\mathfrak{d}' + \mathfrak{c}''\mathfrak{d}' - \mathfrak{c}'\mathfrak{d}''. \tag{54}$$

§ 6. *Seminvariants and semi-covariants.*

Let the system of differential equations (D) be transformed by putting

$$y = \lambda y, \tag{41}$$

where λ is an arbitrary function of u and v. The result of the transformation

is given by equations (42). Those functions of the coefficients of (D), of y, and of the derivatives of these quantities, which are left unchanged by every such transformation, will be called seminvariants or semi-covariants, according as they do not, or do, contain y and its derivatives. It is easily verified that each of the coefficients (45) of the canonical form (E) is a seminvariant, also that the derivatives of these eight quantities are seminvariants. *Every seminvariant of* (D) *is a function of the six quantities* $\mathfrak{b}$, $\mathfrak{b}'$, $\mathfrak{c}'$, $\mathfrak{b}'$, $\mathfrak{c}''$, $\mathfrak{b}''$, *and of their derivatives.* For, in the first place, $\mathfrak{b}$ and $\mathfrak{a}''$ are expressible in terms of $\mathfrak{c}'$ and $\mathfrak{b}'$ respectively, and, in the second place, every equation of form (D) can be reduced to form (E) by a transformation of form (44). But the value of any seminvariant of (D) is the same as for any system, e. g. (E), obtained from (D) by a transformation of form (44), and it therefore reduces to a function of the coefficients of (E).

Since all of the derivatives of y may be expressed in terms of the four quantities y, y_u, y_v, y_{vv}, it is necessary to find only such semi-covariants as involve these four quantities.

The transformation gives

$$\bar{y} = \lambda^{-1} y, \qquad \bar{y}_u = \lambda^{-1}(y_u - \lambda_u \lambda^{-1} y),$$

$$(55)\quad \bar{y}_v = \lambda^{-1}(y_v - \lambda_v \lambda^{-1}) y, \qquad \bar{y}_{vv} = \lambda^{-1}[y_{vv} - 2\lambda_v \lambda^{-1} y_v + (2\lambda_v^2 \lambda^{-2} - \lambda_{vv}\lambda^{-1}) y],$$

while equations (43) give

$$(56)\qquad \bar{e} = e - \lambda_u \lambda^{-1}, \qquad \bar{e}' = e' - \lambda_v \lambda^{-1};$$

whence may be found the following *relative* semi-covariants

$$(57)\qquad y, \qquad \mathfrak{q} = y_u - ey, \qquad \mathfrak{r} = y_v - e' y,$$

$$\mathfrak{s} = y_{vv} - 2e' y_v + (e'^2 - e_v') y = \mathfrak{r}_v - e' \mathfrak{r},$$

which satisfy the equations

$$\bar{y} = \lambda^{-1} y, \qquad \bar{\mathfrak{q}} = \lambda^{-1} \mathfrak{q}, \qquad \bar{\mathfrak{r}} = \lambda^{-1} \mathfrak{r}, \qquad \bar{\mathfrak{s}} = \lambda^{-1} \mathfrak{s}.$$

§ 7. *Invariants and covariants.*

Equation (32) shows that form (D) is left unchanged by any transformation of the form $\bar{u} = \phi(u, v)$, $\bar{v} = \psi(v)$, where ψ is an arbitrary function of v, and ϕ is any function of u and v, which satisfies the differential equation (33). In particular, it will be convenient to make the transformation

$$(58)\qquad \bar{u} = \phi(u), \qquad \bar{v} = \psi(v),$$

where ϕ and ψ are arbitrary functions of the single arguments indicated. It will therefore be assumed that a definite family of plane sections of the

integral surface has been chosen, and that this family is to be preserved in the transformation.

The coefficients of the new equations, obtained by applying the transformation (58) to system (D) are

$$\bar{b}' = \phi_u^{-1}(b - \eta), \quad \bar{d} = \phi_u^{-2} d,$$

$$(59)\quad \bar{b}' = \psi_v^{-1} b', \quad \bar{c}' = \phi_u^{-1} c', \quad \bar{d}' = \phi_u^{-1}\psi_v^{-1} d',$$

$$\bar{a}'' = \psi_v^{-1}(a'' - 3\zeta), \quad \bar{c}'' = \psi_v^{-2}(c'' + a''\zeta - \zeta_v - \zeta^2), \quad \bar{d}'' = \psi_v^{-3} d'',$$

where

$$\eta = \phi_{uu}\phi_u^{-1}, \qquad \zeta = \psi_{vv}\psi_v^{-1}.$$

Those functions of the seminvariants and their derivatives which are left unchanged by a transformation of form (58) are called *invariants*. The seminvariants are all functions of the six quantities $\mathfrak{b}$, $\mathfrak{b}'$, $\mathfrak{c}'$, $\mathfrak{d}'$, $\mathfrak{c}''$, $\mathfrak{d}''$, and their derivatives. The effect of the transformation (58) upon these quantities must, therefore, be investigated. Equations (45) and (59) give

$$(60)\qquad 4\bar{e} = \phi_u^{-1}(4e - \eta), \qquad 4\bar{e}' = \psi_v^{-1}(4e' - 3\zeta),$$

and also,

$$\bar{\mathfrak{b}} = \phi_u^{-2}(\mathfrak{b} + \tfrac{1}{2}\mathfrak{c}'\eta + \tfrac{1}{4}\eta_u - \tfrac{1}{16}\eta^2), \qquad \bar{\mathfrak{b}}' = \psi_v^{-1}(\mathfrak{b}' + \tfrac{3}{4}\zeta),$$

$$\bar{\mathfrak{c}}' = \phi_u^{-1}(\mathfrak{c}' + \tfrac{1}{4}\eta),$$

$$\bar{\mathfrak{b}}' = \phi_u^{-1}\psi_v^{-1}(\mathfrak{d}' - \tfrac{3}{4}\mathfrak{c}'\zeta - \tfrac{1}{4}\mathfrak{b}'\eta - \tfrac{3}{16}\eta\zeta),$$

$$(61)\qquad \bar{\mathfrak{c}}'' = \psi_v^{-2}(\mathfrak{c}'' + \tfrac{1}{2}\mathfrak{b}'\zeta + \tfrac{5}{4}\zeta_v - \tfrac{7}{16}\zeta^2),$$

$$\bar{\mathfrak{d}}'' = \psi_v^{-3}(\mathfrak{d}'' - \tfrac{3}{4}\mathfrak{c}''\zeta - \tfrac{9}{16}\mathfrak{b}'\zeta^2 + \tfrac{3}{4}\mathfrak{b}'\zeta_v + \tfrac{27}{64}\zeta^3 - \tfrac{27}{16}\zeta\zeta_v + \tfrac{3}{4}\zeta_{vv}),$$

$$\bar{\mathfrak{b}}'_v = \psi_v^{-2}(\mathfrak{b}'_v - \mathfrak{b}'\zeta + \tfrac{3}{4}\zeta_v - \tfrac{3}{4}\zeta^2).$$

A sum will be said to be *isobaric* if, in this transformation, each of its terms is multiplied by the same product of the form ϕ_u^λ or ψ_v^μ. Every invariant must be isobaric.

From equations (61), the following relative invariants are found directly

$$(62)\qquad \mathfrak{D}' = \mathfrak{d}' + \mathfrak{b}'\mathfrak{c}', \qquad \mathfrak{C}'' = \mathfrak{c}'' - \tfrac{13}{9}\mathfrak{b}'^2 - \tfrac{5}{3}\mathfrak{b}'_v.$$

If $\mathfrak{D}'$ is different from zero, the further relative invariants

$$(63)\qquad \mathfrak{B}' = \mathfrak{b}' + \tfrac{3}{4}\frac{\mathfrak{D}'_v}{\mathfrak{D}'}, \qquad \mathfrak{C}' = \mathfrak{c}' + \tfrac{1}{4}\frac{\mathfrak{D}'_u}{\mathfrak{D}'},$$

may be obtained. In fact, these four quantities satisfy the equations

$$\bar{\mathfrak{B}}' = \psi_v^{-1}\mathfrak{B}', \qquad \bar{\mathfrak{C}}' = \phi_u^{-1}\mathfrak{C}', \qquad \bar{\mathfrak{D}}' = \phi_u^{-1}\psi_v^{-1}\mathfrak{D}', \qquad \bar{\mathfrak{C}}'' = \psi_v^{-2}\mathfrak{C}''.$$

The seminvariants (45) of system (D) satisfy the integrability conditions of system (E). Equations (50) and (52) together with (62) and (63) show

that it is possible to express the six seminvariants $\mathfrak{d}$, $\mathfrak{b}'$, $\mathfrak{c}'$, $\mathfrak{d}'$, $\mathfrak{c}''$, $\mathfrak{d}''$, and their derivatives in terms of the four invariants $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{D}'$, $\mathfrak{C}''$, and their derivatives. Equations (49), (51), (53), (54) show, moreover, that these four invariants and their derivatives are not independent. In fact, the first three equations give respectively

$$8\,\mathfrak{C}'_v + 4\mathfrak{D}' + 4\mathfrak{B}'_u - 5\frac{\partial^2}{\partial u\,\partial v}\log \mathfrak{D}' = 0,$$

$$\frac{\partial}{\partial u}\log \mathfrak{D}' = 4\mathfrak{C}' + \frac{\partial}{\partial u}\log\,(3\mathfrak{C}'_v - \mathfrak{B}'_u - 6\mathfrak{D}'), \qquad 3\mathfrak{C}'_v - \mathfrak{B}'_u - 6\mathfrak{D}' \neq 0,$$

$$4\mathfrak{B}'\,(13\mathfrak{B}'_u - 39\mathfrak{C}'_v - 42\mathfrak{D}') - 3(13\mathfrak{B}'_u - 39\mathfrak{C}'_v - 42\mathfrak{D})\frac{\partial}{\partial v}\log \mathfrak{D}'$$

$$+ 45\mathfrak{C}''_u - 45\mathfrak{C}'_{vv} + 75\,\mathfrak{B}'_{uv} - 72\,\mathfrak{C}'_v - 36\,\mathfrak{D}' - 36\,\mathfrak{B}'_u - 90\mathfrak{D}'_v = 0.$$

(64)

The condition that the expression $3\mathfrak{C}'_v - \mathfrak{B}'_u - 6\mathfrak{D}'$ shall not vanish, requires, as will be seen, only that the integral surface be not a cone.

Let it be assumed that $\mathfrak{C}''$ is different from zero, and put

$$\mathfrak{E} = \frac{\mathfrak{D}'^2}{\mathfrak{C}''},$$

so that

$$\bar{\mathfrak{E}} = \phi_u^{-2}\mathfrak{E}.$$

Let it also be assumed that $\mathfrak{B}'$ and $\mathfrak{C}'$ are different from zero, and introduce two symbols of operation

$$U = \frac{1}{\mathfrak{C}'}\frac{\partial}{\partial u}, \qquad V = \frac{1}{\mathfrak{B}'}\frac{\partial}{\partial v}.$$

It is clear that

$$U\,(\mathfrak{B}'), \qquad V\,(\mathfrak{C}'), \qquad U\,(\mathfrak{C}''), \qquad V\,(\mathfrak{E}),$$

are invariants. In fact, they satisfy the following equations,

$$\bar{U}\,(\bar{\mathfrak{B}}') = \psi_v^{-1}\,U\,(\mathfrak{C}'), \qquad \bar{V}\,(\bar{\mathfrak{C}}') = \phi_u^{-1}\,V\,(\mathfrak{C}'), \qquad \bar{U}\,(\bar{\mathfrak{C}}'') = \psi_v^{-2}\,U\,(\mathfrak{C}''),$$

$$\bar{V}\,(\bar{\mathfrak{E}}) = \phi_u^{-2}\,V\,(\mathfrak{E}).$$

From any two invariants λ and μ for which $\bar{\lambda} = \phi_u^{-l}\lambda$, $\bar{\mu} = \phi_u^{-m}\mu$, it is always possible to form a new invariant

$$(\mu, \lambda_u) = m\mu\lambda_u - l\lambda\mu_u,$$

their Wronskian with respect to u, which satisfies the equation

$$(\bar{\mu}, \bar{\lambda}_u) = \phi_u^{-l-m}\,(\mu, \lambda_u).$$

By repeating and combining the Wronskian process and the operations U and V, an infinite number of invariants can be deduced from $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{C}''$, and $\mathfrak{E}$. It can now be shown that all invariants are functions of those ob-

tained in this way. It has been seen that the seminvariants (45) and their derivatives may be expressed in terms of $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{C}''$, and $\mathfrak{D}'$. It has also been seen that all invariants are functions of the seminvariants (45) and their derivatives. Therefore any invariant may be expressed as a function of $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{C}''$, $\mathfrak{E}$, and derivatives.

Let the invariants be divided into two classes, of ϕ- and ψ-invariants, according as their transforms are multiplied by powers of ϕ_u or ψ_v respectively. By a process of induction precisely parallel to that employed by Professor Wilczynski, for a similar system of invariants,* it can be shown that there are $n^2 + 2n + 2$ independent (except for equations (54) and (64)) ϕ-invariants of orders equal to or less than n, and the same number of ψ-invariants of the orders named; moreover, that the invariants may be chosen so as to contain all the derivatives of $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{C}''$, $\mathfrak{E}$, up to, and including, those of order n; and, finally, that the two classes of ϕ-invariants and ψ-invariants are independent of each other. If equations (54) and (64) are taken into account, the number of invariants of orders equal to or less than n must be reduced by $\frac{1}{2}(3n^2 + 3n - 4)$. *The Wronskian, the U-process, and the V-process suffice, therefore, to deduce all invariants from the four fundamental ones* $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{C}''$, $\mathfrak{E}$. *The number of independent invariants of order equal to or less than* n *is* $\frac{1}{2}(n^2 + 5n + 12)$, *for* $n \geq 2$. *For* $n = 1$ *there are eight independent invariants.*

Applying the transformation (58) to the semi-covariants (57), we find

$$\bar{y} = y, \qquad \bar{\mathfrak{q}} = \phi_u^{-1}(\mathfrak{q} + \tfrac{1}{4}\eta y), \qquad \bar{\mathfrak{r}} = \psi_v^{-1}(\mathfrak{r} + \tfrac{3}{4}\zeta y), \tag{65}$$

$$\bar{\mathfrak{s}} = \psi_v^{-2}[\mathfrak{s} + \tfrac{1}{2}\mathfrak{r}\zeta + \tfrac{1}{4}(\zeta^2 + 3\zeta_v)\, y];$$

whence we obtain the following relative covariants,

$$y, \qquad \mathfrak{Q} = \mathfrak{q} - \mathfrak{c}'\, y, \qquad \mathfrak{R} = \mathfrak{r} - \mathfrak{b}'\, y, \tag{66}$$

$$\mathfrak{S} = \mathfrak{s} - \tfrac{2}{3}\mathfrak{b}'\,\mathfrak{r} + \tfrac{1}{9}(4\mathfrak{b}'^2 - 9\mathfrak{b}'_v)\, y.$$

They satisfy the equations

$$\bar{y} = y, \qquad \bar{\mathfrak{Q}} = \phi_u^{-1}\mathfrak{Q}, \qquad \bar{\mathfrak{R}} = \psi_v^{-1}\mathfrak{R}, \qquad \bar{\mathfrak{S}} = \psi_v^{-2}\mathfrak{S},$$

and are clearly independent. *All covariants may be expressed in terms of the covariants* (66) *and the invariants.*

§ 8. *Principle of duality. The adjoint system, and its integral curve.*

Given a system of partial differential equations in the second intermediate form (D), for which the conditions (α), (β), (γ) are satisfied, so that it has

* Loc. cit., pp. 254, 255.

four linearly independent analytic solutions

$$y^{(k)} = f^{(k)}(u, v) \qquad (k = 1, 2, 3, 4),$$

which may be interpreted as the homogeneous coördinates of a point y of a developable surface S. The equation of the plane λ, tangent to S at y, is

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y^{(1)} & y^{(2)} & y^{(3)} & y^{(4)} \\ y_u^{(1)} & y_u^{(2)} & y_u^{(3)} & y_u^{(4)} \\ y_v^{(1)} & y_v^{(2)} & y_v^{(3)} & y_v^{(4)} \end{vmatrix} = 0, \tag{67}$$

so that the coördinates of λ are proportional to the minors of x_1, x_2, x_3, x_4, each of which may be represented by an expression of the form

$$\lambda = |\, y,\ y_u,\ y_v \,|, \tag{68}$$

which stands for a determinant of the third order of the matrix

$$||\, y^{(k)},\ y_u^{(k)},\ y_v^{(k)} \,|| \qquad (k = 1, 2, 3, 4)$$

Equation (68), together with (34), (35) and (36), gives

$$\lambda_u = (b + c')\lambda, \tag{69}$$

from which it is possible, by differentiation, to find expressions for all the derivatives of λ taken at least once with respect to u, as $\lambda_{uu}, \lambda_{uv}, \lambda_{uuv}, \lambda_{uvv}$, etc. The following equations hold for the derivatives of λ taken with respect to v alone

$$\begin{aligned} -\lambda_v + b'\lambda + |\, y, y_u, y_{vv} \,| &= 0, \\ (\lambda_{vv}) + 4e' |\, y, y_u, y_{vv} \,| + |\, y_v, y_u, y_{vv} \,| + c' |\, y, y_v, y_{vv} \,| &= 0, \\ (\lambda_{vvv}) + 4(e_v' + 4e'^2) |\, y, y_u, y_{vv} \,| + 8e' |\, y_v, y_u, y_{vv} \,| & \\ + (8e'c' - L) |\, y, y_v, y_{vv} \,| &= 0, \\ (\lambda_{vvvv}) + f |\, y, y_u, y_{vv} \,| + g |\, y_v, y_u, y_{vv} \,| + h |\, y, y_v, y_{vv} \,| &= 0, \end{aligned} \tag{70}$$

where

$$\begin{aligned} (\lambda_{vv}) &= -\lambda_{vv} + b'\lambda_v + (b_v' + c'')\lambda, \\ (\lambda_{vvv}) &= -\lambda_{vvv} + b'\lambda_{vv} + (2b_v' + c'')\lambda_v + (b_{vv}' + c_v'' + 4e'c'' - d'')\lambda, \\ (\lambda_{vvvv}) &= -\lambda_{vvvv} + b'\lambda_{vvv} + (3b_v' + c'')\lambda_{vv} + (3b_{vv}' + 2c_v'' + 4e'c'' - d'')\lambda_v \\ &\quad + [(b_{vv}' + c_v'' + 4e'c'' - d'')_v + 4c''(e_v' + 4e'^2) - 8e'd'']\lambda, \\ f &= 4(e_v' + 4e'^2)_v + 16e'(e_v' + 4e'^2), \qquad g = +12e_v' + 48e'^2, \\ h &= 4e'(4e'c' - 2d') + 4c'e_v' + (8e'c' - L)_v + a''(8e'c' - L). \end{aligned}$$

If the invariant expression

$$L = b'c' + d' - c'_v = b'\mathfrak{c}' + \mathfrak{b}' - \mathfrak{c}'_v = -\tfrac{1}{6}(3\mathfrak{C}'_v - \mathfrak{B}'_u - 6\mathfrak{D}')$$

is different from zero, the determinants $|y, y_u, y_{vv}|$, $|y_v, y_u, y_{vv}|$, $|y, y_v, y_{vv}|$, may be eliminated from the above equations so as to obtain an equation of the fourth order for the coördinates of the plane λ,

$$(71)\qquad \begin{vmatrix} -\lambda_v + b'\lambda & 1 & 0 & 0 \\ (\lambda_{vv}) & 4e' & 1 & c' \\ (\lambda_{vvv}) & 4(e'_v + 4e'^2) & 8e' & 8e'c' - L \\ (\lambda_{vvvv}) & f & g & h \end{vmatrix} = 0,$$

say, for brevity,

$$(71)\qquad L\lambda_{vvvv} + L_1\lambda_{vvv} + L_2\lambda_{vv} + L_3\lambda_v + L_4\lambda = 0.$$

The completely integrable system, composed of equations (69) and (71), which may be called the adjoint system of (D), represents the same integral surface as system (D), but in plane- instead of point-coördinates. The planes whose coördinates are $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)}$ are the tangent planes of the surface, which, in general, change with v. As u changes, each plane remains fixed. Equation (71) may be thought of as the differential equation, in plane coördinates, of the edge of regression, C, of the integral developable surface S of system (D). As v changes, the plane λ envelops the curve C. As u changes, the plane remains fixed, corresponding to the fact that at all points of one and the same generator the developable has the same tangent plane.

If L is identically zero, $|y, y_u, y_{vv}|$, $|y_v, y_u, y_{vv}|$, $|y, y_v, y_{vv}|$ may be eliminated from the first three of the equations (70), so as to obtain an equation of the third order for the coördinates of the plane λ,

$$(72)\qquad \begin{aligned} &\lambda_{vvv} + \lambda_{vv}(-8e' - b') + \lambda_v(8b'e' + 16e'^2 - 4e'_v - 2b'_v - c'') \\ &\quad + \lambda(4e'c'' + 8e'b'_v + 4e'_v b' - 16b'e'^2 - b'_{vv} - c''_v + d'') = 0. \end{aligned}$$

In this case, equations (69) and (72) show that the developable degenerates into a cone, since the coördinates of its tangent planes must satisfy a linear homogeneous relation with constant coefficients. In fact, equation (72) shows that $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)}$ satisfy a relation of the form

$$\phi_1(u)\lambda^{(1)} + \phi_2(u)\lambda^{(2)} + \phi_3(u)\lambda^{(3)} + \phi_4(u)\lambda^{(4)} = 0.$$

But according to (69), the ratios of $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)}$ are independent of u, so that such a relation would reduce to one with constant coefficients. Consequently all of the tangent planes of the developable have a point in common, that is, the cuspidal edge of the developable degenerates into a

point, and the developable is necessarily a cone. Therefore *if the invariant L vanishes identically the integral surface of system (D) is a cone.*

If the original system is in the canonical form, so that $e' = 0$, the adjoint system has the comparatively simple form

$$\lambda_u + \mathfrak{c}'\lambda = 0,$$

$$\begin{aligned} &L\lambda_{vvvv} - L_v\lambda_{vvv} + [\mathfrak{b}' L_v - L(3\mathfrak{b}'_v + \mathfrak{c}'' + \mathfrak{b}'^2)]\lambda_{vv} \\ &\quad + [(L_v - \mathfrak{b}' L)(2\mathfrak{b}'_v + \mathfrak{c}'') - L(3\mathfrak{b}'_{vv} + 2\mathfrak{c}''_v - \mathfrak{b}'')]\lambda_v \\ &\quad + [(L_v - \mathfrak{b}' L)(\mathfrak{b}'_{vv} + \mathfrak{c}''_v - \mathfrak{b}'') - L(\mathfrak{b}'_{vvv} + \mathfrak{c}''_{vv} - \mathfrak{b}''_v)]\lambda = 0. \end{aligned} \tag{73}$$

§ 9. *The transformation of Laplace. The cuspidal edge of the integral developable.*

Consider any surfaces S_0, not necessarily developable. We shall assume that its homogeneous point-coördinates, $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$, are functions of two parameters, u and v, chosen so that the two families of curves, $u = \text{const.}$ and $v = \text{const.}$, on it, form a conjugate system. DARBOUX has shown* that this condition is fulfilled if, and only if, the four functions $y^{(k)}$ satisfy an equation of the form

$$y_{uv} = A(u, v)\, y_u + B(u, v)\, y_v + C(u, v)\, y. \tag{74}$$

The tangents to the family of curves $u = \text{const.}$, at the points y, form a congruence Γ_1. One sheet of the focal surface of this congruence is the given surface S_0. The other sheet is the surface S_1 whose coördinates $y_1^{(k)}$ are given by the Laplace transform of y with respect to v,

$$y_1 = y_v - Ay.$$

Indeed, it is easily verified that y_1 satisfies an equation of the form

$$(y_1)_u - By = Dy,$$

which shows that any line tangent to S_0 at y, along a curve $u = \text{const.}$ on S_0, is also tangent to S_1 at y_1, along a curve $v = \text{const.}$ on S_1.

Similarly the tangents to the family of curves $v = \text{const.}$, on the given surface S_0, form a congruence Γ_{-1} such that the coördinates of the second sheet S_{-1} of its focal surface are given by the Laplace transform of y with respect to u,

$$y_{-1} = y_u - By.$$

The Laplace transformation may be repeated indefinitely, so that, in general, there is obtained an unlimited sequence of surfaces, S_k ($k = \cdots -3, -2, -1, 0, 1, 2, 3, \cdots$), on each of which the curves $u = \text{const.}$ and $v = \text{const.}$ are conjugate families; while to each surface S_k corresponds a

* DARBOUX, *Leçons sur la théorie générale des surfaces*, vol. I, p. 122.

pair of congruences Γ_{k+1}, Γ_{k-1}, formed by the tangents to the curves $u=$const., $v=$ const., respectively.

The case in which one or more of the surfaces S_k are developables is of importance for our theory. We have seen that the differential equations of any developable surface may be written in the form (B). The two families of curves, $u=$ const., $v=$ const. are still perfectly arbitrary. They form a conjugate system if, and only if, a' vanishes identically. But when a' equals zero, system (B) reduces to its first intermediate form (C), and one of the conjugate families necessarily consists of the generators. The form of system (C) shows that this is the family $v=$ const. The other family is left entirely arbitrary. In this case the relation of conjugacy between the two families of curves becomes degenerate.

We now see that the covariant

$$\Omega = y_u - c' y, \tag{66b}$$

of system (C) is the Laplace transform of y with respect to u.

If four linearly independent solutions $y^{(k)}$ $(k = 1, 2, 3, 4)$, of system (C) be put for y into this expression, and also into Ω_v, there will be found two points. The second equation of system (C) gives the following relation between these points,

$$\Omega_v = b' y_u + y (d' - c'_v). \tag{75}$$

If u is constant, while v varies continuously, Ω describes, in general, a space curve $u=$ const. Ω_v is a point on the tangent to this curve at the point Ω. Equation (75) shows that Ω_v is also a point of the generator (of the surface S_0) containing the points y, y_u, Ω. *The point Ω is, therefore, that point of the generator of the developable S_0 in which it intersects the cuspidal edge.* The ratios of $\Omega^{(1)}$, $\Omega^{(2)}$, $\Omega^{(3)}$, $\Omega^{(4)}$ must, accordingly, be independent of u.

The differential equation which characterizes the cuspidal edge of the integral developable S_0 of system (C), considered as the locus of the point Ω may be found as follows. Multiplying equation (66b) by b' and subtracting from equation (75) we find

$$\Omega_v - b' \Omega = Ly, \tag{76}$$

where

$$L = b' c' + d' - c'_v,$$

is the invariant discussed in § 7.

If L vanishes identically, equation (76) shows that the ratios of $\Omega^{(k)}$ are independent not only of u, but also of v, that is, they are constants; and the cuspidal edge is a point, as should be expected, since the integral surface is a cone.

If L is not identically zero, differentiation of equation (76) gives

$$\begin{aligned}
&\Omega_{vv} - b'\Omega_v - b'_v\Omega = L_v y + L y_v,\\
(77)\qquad &\Omega_{vvv} - b'\Omega_{vv} - 2b'_v\Omega_v - b'_{vv}\Omega = L_{vv} y + 2L_v y_v + L y_{vv},\\
&\Omega_{vvvv} - b'\Omega_{vvv} - 3b'_v\Omega_{vv} - 3b'_{vv}\Omega_v - b'_{vvv}\Omega = (L_{vvv} + d''L)\, y\\
&\qquad\qquad + (3L_{vv} + c''L)\, y_v\\
&\qquad\qquad + (3L_v + a''L) y_{vv} + b''L y_u.
\end{aligned}$$

When y_u is replaced by its equal, $\Omega + c'y$, the expressions y, y_v, y_{vv}, may be eliminated from the four equations (76), (77), so that the desired equation is found in the form

$$(78)\qquad E_0\,\Omega_{vvvv} + E_1\,\Omega_{vvv} + E_2\,\Omega_{vv} + E_3\,\Omega_v + E_4\,\Omega = 0.$$

The projective properties of the cuspidal edge of the integral surface S_0 of system (C) may be studied directly from this equation. The projective theory of space curves, considered from this point of view, has already been developed by Professor Wilczynski.* We shall try, therefore, merely to find the relations between the invariants and covariants of this differential equation and the invariants and covariants of system (C).

For this purpose, we shall assume that system (C) has been reduced to its canonical form (E). Its coefficients, then, are seminvariants, which remain unchanged under any transformation of the form

$$\lambda(u, v)\, y = \bar{y}.$$

Therefore the equation corresponding to (78) may be obtained by substituting the expression (76) for y into equation (48) of system (E). This gives for the equation of the cuspidal edge,

$$\begin{aligned}
[L'(\Omega_v - b'\Omega)]_{vvv} = &- b'[L^{-1}(\Omega_v - b'\Omega)]_{vv}\\
&+ c''[L^{-1}(\Omega_v - b'\Omega)]_v + b''L^{-1}(\Omega_v - b'\Omega),
\end{aligned}$$

or

$$\Omega_{vvvv} + 4p_1\Omega_{vvv} + 6p_2\Omega_{vv} + 4p_3\Omega_v + p_4\Omega = 0,$$

where, in particular, $p_1 = -\frac{3}{4}L_v L^{-1}$.

By making the transformation

$$\Omega = L^{\frac{1}{2}}\bar{\Omega}$$

this equation may be reduced to its *semi-canonical* form

$$(79)\qquad \bar{\Omega}_{vvvv} + 6\pi^{(2)}\bar{\Omega}_{vv} + 4\pi^{(3)}\bar{\Omega}_v + \pi^{(4)}\bar{\Omega} = 0,$$

which is characterized by the absence of the term involving the third derivative.

* E. J. Wilczynski, *Projective differential geometry of curves and ruled surfaces*, Teubner, Leipzig, 1906, Chapter XIII.

The remaining coefficients will be

$$\pi^{(2)} = l^{-1}[\,3l_{vv} + 3m_v + \mathfrak{b}'\,(2l_v + m) - \mathfrak{c}''\,l\,],$$

$$\pi^{(3)} = l^{-1}[\,l_{vvv} + 3m_{vv} + \mathfrak{b}'\,(l_{vv} + 2m_v) - \mathfrak{c}''\,(l_v + m) - \mathfrak{b}''\,l],$$

$$\pi^{(4)} = l^{-1}[\,m_{vvv} + \mathfrak{b}'\,m_{vv} - \mathfrak{c}''\,m_v - \mathfrak{b}''\,m\,],$$

where

$$l = L^{-\frac{1}{2}}, \qquad m = \tfrac{3}{4}l^5\,L_v - \mathfrak{b}'\,l.$$

From these expressions for the coefficients a complete system of invariants and covariants may be readily computed. For example,* we have

$$\Theta_3 = \pi^{(3)} - 3\pi_v^{(2)}.$$

If Θ_3 is identically equal to zero, the generators of S_0 belong to a linear complex.†

Consider, next, the covariant

$$\mathfrak{R} = y_v - b'\,y, \tag{66c}$$

of system (C) which was found in § 7. It is the Laplace transform of y with respect to v. Therefore, if y is interpreted as a point on the integral surface S_0, $\mathfrak{R}$ must be interpreted as a point on the surface S_{-1} which constitutes the second sheet of the congruence Γ_{-1} formed by the tangents to the curves $u = \text{const.}$ on S_0. The differential equations for the surface S_{-1} may be found as follows.

The second equation of system (C) enables us to write

$$\mathfrak{R}_u = c'\,y_v + (d' - b'_u)\,y. \tag{80}$$

Multiplying (66c) by c' and subtracting from (80), we obtain,

$$\mathfrak{R}_u - c'\,\mathfrak{R} = My, \tag{81}$$

where

$$M = b'\,c' + d' - b'_u = 4\mathfrak{D}' - 2L = \tfrac{2}{5}\,(4\mathfrak{D}' + 3\mathfrak{C}'_v - \mathfrak{B}'_u).$$

If the invariant M vanishes identically, equation (81) shows that *the surface S_{-1} degenerates into a curve.*

If M is not identically equal to zero, the desired equations may be found by eliminating y, y_u, y_v, y_{vv} from equations (66c), (81), and others derived from them by differentiation. We thus obtain a system of form (1)

$$\mathfrak{R}_{uv} = c'\,\mathfrak{R}_v + \left(b' + \frac{M_v}{M}\right)\mathfrak{R}_u + \left[M + c'_v - c'\left(b' + \frac{M_v}{M}\right)\right]\mathfrak{R},$$

* Ibid., p. 240.

† Ibid., p. 254.

$$(82)\qquad \begin{vmatrix} \mathfrak{R}_{vv} & a''-b' & b'' & c''-2b'_v & d''-b'_{vv} \\ \mathfrak{R}_v & 1 & 0 & -b' & -b'_v \\ \mathfrak{R} & 0 & 0 & 1 & -b' \\ \mathfrak{R}_u - c'\mathfrak{R} & 0 & 0 & 0 & M \\ \mathfrak{R}_{uu} - c'\mathfrak{R}_u - c'_u\mathfrak{R} & 0 & M & 0 & M_u \end{vmatrix} = 0,$$

for which the coefficients of the terms of the second order are

$$A = C = B' = 0, \qquad B = 1, \qquad A' = b''M, \qquad C' = M^2,$$

so that the invariant K has the value $b''M^3$. The integral surface is developable if, and only if, b'' vanishes identically.

Therefore, *if a one-parameter family of curves, C, be drawn upon a developable, the tangents to these curves form, of course, a congruence. The second sheet of the focal surface of this congruence will degenerate into a curve if, and only if, M is equal to zero. If it does not degenerate, it will itself be a developable if, and only if, the curves C are plane curves.*

§ 10. *Canonical developments of the non-homogeneous coördinates of a developable surface.*

The semi-covariants of system (E) were found to be

$$y, \qquad \mathfrak{q} = y_u, \qquad \mathfrak{r} = y_v, \qquad \mathfrak{s} = y_{vv}.$$

If four linearly independent solutions, $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$, be substituted for y in these expressions, three points, $\mathfrak{q}, \mathfrak{r}, \mathfrak{s}$, semi-covariantly related to the point y, will be found. These points, together with y, form a non-degenerate tetrahedron, except for certain singular points. This tetrahedron will be used as a tetrahedron of reference, and, more specifically, the coördinates of any point with respect to it will be defined in the following way. If we substitute into any expression of the form

$$\lambda = \alpha\mathfrak{s} + \beta\mathfrak{q} + \gamma\mathfrak{r} + \delta y,$$

for y the four linearly independent solutions, $y^{(1)}, y^{(2)}, y^{(3)}, y^{(4)}$, we find four functions, $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)}$, the homogeneous coördinates of a point λ. The coördinates of the point λ referred to the tetrahedron of reference, $y, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}$, are then defined by the equations

$$x_1 = \alpha, \qquad x_2 = \beta, \qquad x_3 = \gamma, \qquad x_4 = \delta.$$

We shall need expressions for the derivatives of y up to and including the

sixth order. We find, for example,

$$
\begin{aligned}
&y_u = \mathfrak{q}, \qquad y_v = \mathfrak{r},\\
&y_{uu} = -2\mathfrak{c}'\,\mathfrak{q} + \mathfrak{b}\,y, \qquad y_{uv} = \mathfrak{b}'\,\mathfrak{q} + \mathfrak{c}'\,\mathfrak{r} + \mathfrak{b}'\,y, \qquad y_{vv} = \mathfrak{s},\\
&y_{uuu} = (7\mathfrak{c}'^2 - \mathfrak{c}_u)\,\mathfrak{q} + (\mathfrak{b}_u - 2\mathfrak{c}'\,\mathfrak{b})\,y,\\
&y_{uuv} = -2(\mathfrak{c}'_v + \mathfrak{c}'\,\mathfrak{b}')\,\mathfrak{q} + (\mathfrak{b} - 2\mathfrak{c}'^2)\,\mathfrak{r} + (\mathfrak{b}_v - 2\mathfrak{c}'\,\mathfrak{b}')\,y,\\
&y_{uvv} = \mathfrak{c}'\,\mathfrak{s} + (\mathfrak{b}'_v + \mathfrak{b}'^2)\,\mathfrak{q} + (\mathfrak{c}'_v + \mathfrak{b}' + \mathfrak{b}'\,\mathfrak{c}')\,\mathfrak{r} + (\mathfrak{b}'_v + \mathfrak{b}'\,\mathfrak{b}')\,y,\\
&y_{vvv} = -\mathfrak{b}'\,\mathfrak{s} + \mathfrak{c}''\,\mathfrak{q} + \mathfrak{b}''\,y.
\end{aligned}
$$

All such expressions are of the form

$$y_{u_i v_j} = A^{ij}\,\mathfrak{s} + B^{ij}\,\mathfrak{q} + C^{ij}\,\mathfrak{r} + D^{ij}\,y,$$

where the subscript $u_i v_j$ indicates $i+j$ differentiations, i with respect to u and j with respect to v. The explicit expressions for the coefficients A^{ij}, B^{ij}, $\cdots$, will be omitted for $i+j>3$, since they are rather long, and since their calculation is of little interest and offers no difficulty.

We shall assume that the point y of the surface corresponds to the value $(0, 0)$ of u and v, an assumption which involves no loss of generality. If the surface is analytic in the vicinity of this point, for values of u and v sufficiently small, the general solution of system (E) may be expanded into a power series

$$Y = y + y_u\,u + y_v\,v + \tfrac{1}{2}(y_{uu}\,y^2 + 2y_{uv}\,uv + y_{vv}\,v^2) + \cdots,$$

which may be written in the form

$$Y = t_1\,\mathfrak{s} + t_2\,\mathfrak{q} + t_3\,\mathfrak{r} + t_4\,y.$$

The coefficients, t_1, t_2, t_3, t_4, are power series, which, in accordance with our convention, represent the coördinates of any point of the surface, within a certain vicinity of the point y, referred to the tretahedron $y, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}$. These series are

$$
\begin{aligned}
t_1 &= \tfrac{1}{2}v^2 + \tfrac{1}{2}\mathfrak{c}'\,uv^2 - \tfrac{1}{6}\mathfrak{b}'\,v^3 + \frac{1}{4!}A^{40}\,u^4 + \cdots,\\
t_2 &= u - \mathfrak{c}'\,u^2 + \mathfrak{b}'\,uv + \tfrac{1}{6}(7\mathfrak{c}'^2 - \mathfrak{c}'_u)\,u^3 - (\mathfrak{c}'_v + \mathfrak{b}'\,\mathfrak{c}')\,u^2 v\\
&\qquad + \tfrac{1}{2}(\mathfrak{b}'_v + \mathfrak{b}'^2)uv^2 + \frac{1}{4!}B^{40}\,u^4 + \cdots,\\
t_3 &= v + \mathfrak{c}'\,uv + \tfrac{1}{2}(\mathfrak{c}'_u + \mathfrak{c}'^2)\,u^2 v + \tfrac{1}{2}(\mathfrak{c}'_v + \mathfrak{b}' + \mathfrak{b}'\,\mathfrak{c}')\,uv^2 \qquad (83)\\
&\qquad + \tfrac{1}{6}\mathfrak{c}''\,v^3 + \frac{1}{4!}C^{40}\,u^4 + \cdots,\\
t_4 &= 1 + \tfrac{1}{2}\mathfrak{b}\,u^2 + \mathfrak{b}'\,uv + \tfrac{1}{6}(\mathfrak{b}_u - 2\mathfrak{c}'\,\mathfrak{b})\,u^3 + \tfrac{1}{2}(\mathfrak{b}_v - 2\mathfrak{c}'\,\mathfrak{b}')\,u^2 v\\
&\qquad + \tfrac{1}{2}(\mathfrak{b}'_v + \mathfrak{b}'\,\mathfrak{b}')\,uv^2 + \tfrac{1}{6}\mathfrak{b}''\,v^3 + \frac{1}{4!}D^{40}\,u^4 + \cdots.
\end{aligned}
$$

Let non-homogeneous coördinates be introduced by putting

$$\zeta = 2\frac{t_1}{t_4}, \qquad \eta = \frac{t_2}{t_4}, \qquad \xi = \frac{t_3}{t_4}, \tag{84}$$

so that the development of the surface, exact up to terms of the sixth order, takes the form

$$\zeta = \xi^2 + \phi_3(\xi,\eta) + \phi_4(\xi,\eta) + \phi_5(\xi,\eta) + \phi_6(\xi,\eta) + \cdots, \tag{85}$$

where

$$\phi_3(\xi,\eta) = \zeta_{03}\xi^3 + \zeta_{12}\eta\xi^2 + \zeta_{21}\eta^2\xi + \zeta_{30}\eta^3, \text{ etc.}$$

The explicit expressions for these coefficients ζ_{ij}, which do not vanish identically, are

$$\begin{aligned}
&\zeta_{12} = -\mathfrak{c}', \qquad \zeta_{03} = -\tfrac{1}{3}\mathfrak{b}', \qquad \zeta_{22} = \mathfrak{c}'^2,\\
&\zeta_{13} = -\tfrac{1}{3}(\mathfrak{b}'_u + 3\mathfrak{c}'_v - 2\mathfrak{b}'\mathfrak{c}'), \qquad \zeta_{04} = \tfrac{1}{12}(\mathfrak{b}'^2 - \mathfrak{b}'_v - 3\mathfrak{c}''),\\
&\zeta_{32} = -\mathfrak{c}'^3, \qquad \zeta_{23} = \mathfrak{c}'(\mathfrak{b}'_u - \mathfrak{b}'\mathfrak{c}' + 3\mathfrak{c}'_v),\\
&\zeta_{14} = \tfrac{1}{12}(16\mathfrak{b}'\mathfrak{c}'_v + 3\mathfrak{c}'\mathfrak{b}'_v - \mathfrak{b}'^2\mathfrak{c}' + 2\mathfrak{b}'\mathfrak{b}' - 3\mathfrak{c}'_{vv} - \mathfrak{b}'_{uv} + 6\mathfrak{b}'\mathfrak{b}'_u + 9\mathfrak{c}'\mathfrak{c}''),\\
&\zeta_{05} = \tfrac{1}{12}(25\mathfrak{b}'' + 47\mathfrak{c}''_v - 45\mathfrak{b}'\mathfrak{c}'' - 24\mathfrak{b}'_{vv} + 72\mathfrak{b}'\mathfrak{b}'_v - 24\mathfrak{b}'^3),\\
&\zeta_{42} = \mathfrak{c}'^4, \qquad \zeta_{33} = \tfrac{2}{3}\mathfrak{c}'^2(2\mathfrak{b}'\mathfrak{c}' - 9\mathfrak{c}'_v - 3\mathfrak{b}'_u),\\
&\zeta_{24} = \tfrac{1}{12}(-2\mathfrak{b}'^2\mathfrak{c}'^2 - 6\mathfrak{c}'^2\mathfrak{b}'_v - 18\mathfrak{c}''\mathfrak{c}'^2 - 64\mathfrak{b}'\mathfrak{c}'\mathfrak{c}'_v - 8\mathfrak{b}'\mathfrak{c}'\mathfrak{b}' + 12\mathfrak{c}'\mathfrak{c}'_{vv}\\
&\qquad\qquad + 4\mathfrak{c}'\mathfrak{b}'_{uv} - 24\mathfrak{b}'_u\mathfrak{b}'\mathfrak{c}' + 3\mathfrak{b}'^2_u + 27\mathfrak{c}'^2_v + 18\mathfrak{b}'_u\mathfrak{c}'_v),\\
&\zeta_{15} = \tfrac{1}{60}(6\mathfrak{b}''_u - 3\mathfrak{c}'''_{uv} + 13\mathfrak{b}'_u\mathfrak{c}'' + 13\mathfrak{b}'\mathfrak{c}''_u - \mathfrak{b}'_{uvv} + 8\mathfrak{b}'\mathfrak{b}'_{uv} + 13\mathfrak{b}'_u\mathfrak{b}'_v\\
&\qquad -23\mathfrak{b}'^2\mathfrak{b}'_u - 619\mathfrak{c}'\mathfrak{b}'' - 1{,}178\mathfrak{c}'\mathfrak{c}''_v + 1{,}138\mathfrak{b}'\mathfrak{c}'\mathfrak{c}'' + 599\mathfrak{c}'\mathfrak{b}'_{vv} - 1{,}827\mathfrak{c}'\mathfrak{b}'\mathfrak{b}'_v\\
&\qquad + 549\mathfrak{b}'^3\mathfrak{c}' - 506\mathfrak{b}'^2\mathfrak{b}' - 306\mathfrak{b}'\mathfrak{b}'_v + 45\mathfrak{c}'_v\mathfrak{c}'' + 45\mathfrak{c}'_v\mathfrak{b}'_v - 75\mathfrak{c}'_v\mathfrak{b}'^2 + 15\mathfrak{b}'\mathfrak{c}'_{vv}),\\
&\zeta_{06} = \tfrac{1}{360}(-84\mathfrak{b}''_v + 84\mathfrak{b}'\mathfrak{b}'' - 3\mathfrak{c}''_{vv} - 28\mathfrak{b}'^2\mathfrak{c}'' + 45\mathfrak{c}''^2 - \mathfrak{b}'_{vvv}\\
&\qquad\qquad + 3\mathfrak{b}'^2_v + \mathfrak{b}'^4 + 28\mathfrak{b}'_v\mathfrak{c}'' + 6\mathfrak{b}'\mathfrak{c}''_v - 4\mathfrak{b}'^2\mathfrak{b}'_v + 4\mathfrak{b}'\mathfrak{b}'_{vv}).
\end{aligned} \tag{86}$$

Since the surface is developable, the equation

$$\left(\frac{\partial^2\zeta}{\partial\xi\,\partial\eta}\right)^2 - \frac{\partial^2\zeta}{\partial\xi^2}\frac{\partial^2\zeta}{\partial\eta^2} = 0,$$

must be satisfied identically, so that the following relations among the coefficients must hold,

$$\begin{aligned}
&\zeta_{22} = \zeta_{12}^2, \qquad \zeta_{23} = 3\zeta_{12}(\zeta_{13} - \zeta_{03}\zeta_{12}), \qquad \zeta_{32} = \zeta_{12}^3,\\
&4\zeta_{24} = 12\zeta_{12}^2(3\zeta_{03}^2 - 2\zeta_{04}) + 4\zeta_{12}(4\zeta_{14} - 9\zeta_{03}\zeta_{12}) + 9\zeta_{13}^2,\\
&\zeta_{33} = 6\zeta_{12}^2\zeta_{13} - 8\zeta_{12}^3\zeta_{03}, \qquad \zeta_{42} = \zeta_{12}^4.
\end{aligned} \tag{87}$$

These equations will be used repeatedly in what follows, without further reference to them.

Every term in the right member of equation (85) is divisible by ξ^2, that is, the plane $\zeta=0$ is tangent to the surface at all points of the generator $\xi=\zeta=0$. The most general transformation of coördinates which leaves this plane and this generating line unchanged is

$$\xi = \lambda \frac{a'x + c'z}{1 + ax + by + cz}, \qquad \eta = \mu \frac{a''x + b''y + c''z}{1 + ax + by + cz},$$

$$\zeta = \frac{\nu z}{1 + ax + by + cz}, \tag{88}$$

where $\lambda, \mu, \nu, a, b, \cdots c''$, are arbitrary constants, which may be chosen in such a way as to simplify the development into series.*

If, instead of y, the linear combination $a''x + b''y$ is taken, as Darboux has suggested, the transformation (88) may be replaced by the following three. First put

$$\xi = \frac{x + hz}{1 + u_1 + u_0 z}, \qquad \eta = \frac{y + kz}{1 + u_1 + u_0 z}, \qquad \zeta = \frac{z}{1 + u_1 + u_0 z}, \tag{89}$$

where u_1 is a linear function of x and y, say,

$$u_1 = Ax + By,$$

and u_0 is a constant. Follow this transformation by a linear transformation of x and y,

$$x = a'x', \qquad y = a''x' + b''y'; \tag{90}$$

and finally by a transformation of the form

$$\xi = \lambda x, \qquad \eta = \mu y, \qquad \zeta = \nu z. \tag{91}$$

If the values of ξ, η, ζ, given by equations (89), be substituted into the development (85), it becomes

$$z = \frac{(x + hz)^2}{1 + u_1 + u_0 z} + \frac{\phi_3 (x + hz, \quad y + kz)}{(1 + u_1 + u_0 z)^2} + \cdots.$$

This, of course, must be of the form

$$z = x^2 + U_3 + U_4 + \cdots, \tag{92}$$

and, up to terms of the fourth order, the following equation must hold

$$z = (x + hz)^2 (1 - u_1 - u_0 z + u_1^2) + \phi_3 (x, y) (1 - 2u_1) + z\Delta\phi_3 + \phi_4,$$

where Δ represents the operation $h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}$. Replacing z by the develop-

* The development to terms of the fifth order inclusive is given by Darboux, Bulletin des Sciences mathématiques, 2d series, vol. 4, 1880, p. 356.

ment (92), and equating the terms of the same degree, we find

$$U_3 = -u_1 x^2 + 2hx^3 + \phi_3,$$

$$U_4 = -u_0 x^4 + u_1^2 x^2 - hu_1 x^3 + 2hxU_3 - 2u_1 \phi_3 + x^2 \Delta\phi_3 + \phi_4.$$

The two constants involved in u_1, may be chosen so as to make U_3 vanish identically. To accomplish this, it is necessary and sufficient that they satisfy the equations

$$2h - A + \zeta_{03} = 0, \qquad -B + \zeta_{12} = 0.$$

If these conditions are satisfied the terms of the fourth order become

$$U_4 = (-u_0 - A^2 + \zeta_{04} + 3h\zeta_{03} + k\zeta_{12})x^4 + (\zeta_{13} - 2\zeta_{12}\zeta_{03})x^3 y.$$

Let us suppose that U_3 has been reduced to zero in the way indicated, so that for the new development

$$\phi_3 = \phi_3' = 0, \qquad \phi_4 = \phi_4'.$$

In order that U_3 may remain zero for any subsequent transformation of form (89), it is only necessary to have

$$2hx - u_1 = 0, \qquad \text{that is,} \qquad A = 2h, \qquad B = 0.$$

Upon making such a transformation, the new U_4 becomes

$$U_4 = -u_0 x^4 + \phi_4' = (\zeta_{04}' - u_0) x^4 + (\zeta_{13} - 2\zeta_{12}\zeta_{03}) x^3 y.$$

By putting $u_0 = \zeta_{04}'$ it is, therefore, possible to reduce the development to the form

$$z = x^2 + \Delta_{13} x^3 y + \phi_5(x, y) + \phi_6(x, y) + \cdots,$$

where

$$\Delta_{13} = \zeta_{13} - 2\zeta_{12}\zeta_{03} = \tfrac{1}{3}L,$$

L being the invariant found in § 8.

The development may be further simplified by making another transformation of the form (89), viz.,

$$x = \frac{x' + hz'}{1 + 2hx'}, \qquad y = \frac{y' + kz'}{1 + 2hx'}, \qquad z = \frac{z'}{1 + 2hx'}, \tag{93}$$

where we have taken

$$u_0 = 0, \qquad B = 0, \qquad A = 2h,$$

so as not to disturb the form of development already established. The new development becomes

$$\begin{aligned} z' &= \frac{(x' + hz')^2}{1 + 2hx'} + \Delta_{13} \frac{(x' + hz')^3 (y' + kz')}{(1 + 2hx')^2} + \frac{\phi_5(x' + hz', y' + kz')}{(1 + 2hx')^4} + \cdots \\ &= (x' + hz')^2 (1 - 2hx' + 4h^2 x'^2 - 8h^3 x'^3 + 16h^4 x'^4) \\ &\quad + \Delta_{13} (x'^3 + 3hx'^2 z' + 3h^2 x' z'^2)(y' + kz')(1 + 12h^2 x'^2 - 4hx') \\ &\quad + \phi_5(x', y')(1 - 8hx') + z' \Delta\phi_5 + \phi_6 + \cdots \\ &= x'^2 + \Delta_{13} x'^3 y' + U_5 + U_6 + \cdots, \end{aligned}$$

where

$$U_5 = x'^5 (k\Delta_{13} - 2h^3) + h\Delta_{13} x'^4 y' + \phi_5 (x', y')$$
$$= x'^5 (k\Delta_{13} - 2h^3 + z'_{05}) + x'^4 y' (h\Delta_{13} + z'_{14}).$$

The coefficient of x'^5 may be made equal to zero by putting

$$k\Delta_{13} - 2h^3 + z'_{05} = 0.$$

The term in $x'^4 y'$ also may be reduced to zero, by choice of h, provided Δ_{13} does not vanish identically,that is, whenever the surface is not a cone.

Suppose that the surface is not a cone, and let the terms of the fifth order be removed, as indicated. U_5 will remain zero under subsequent transformations of the form considered, only provided $h = k = 0$, that is, only if the transformation is the identical transfomation. The terms of the sixth order become

$$U_6 = z'_{06} x'^6 + z'_{15} x'^5 y' + \tfrac{9}{4}\Delta_{13}^2 x'^4 y'^2;$$

and, by putting,

$$y'' = \Delta_{13} y',$$

the series reduces to the form

$$z' = x'^2 + x'^3 y'' + z'_{06} x'^6 + z'_{15} x'^5 y'' + \tfrac{9}{4} x'^4 y''^2 + \cdots.$$

It will now be convenient to introduce the linear transformation given by equations (90). But, in order to preserve the form of the development just found, it will be necessary to employ a combination of the transformations (89) and (90). Since all of the terms of this development are of even order, it is clear that the most general transformation which need be considered is one of the form

$$x' = \frac{a' X}{1 + cZ}, \qquad y'' = \frac{a'' X + b'' Y}{1 + cZ}, \qquad z' = \frac{Z}{1 + cZ}.$$

When such a transformation is made, the development becomes

$$Z = a'^2 X^2 (1 - cZ + c^2 Z^2) + a'^3 X^3 (1 - cZ)^3 (a'' X + b'' Y) + z'_{06} a'^6 X^6$$
$$+ z'_{15} a'^5 X^5 (a'' X + b'' Y) + \tfrac{9}{4} a'^4 X^4 (a'' X + b'' Y)^2 + \cdots.$$

In order that this may again assume the form

$$Z = X^2 + X^3 Y + U_6 + \cdots,$$

it is necessary to have

$$a'^2 = 1, \qquad -a'^2 c + a'^3 a'' = 0, \qquad b'' a'^3 = 1.$$

Assuming these equations to be satisfied, the terms of the sixth order become

$$U_6 = (Z_{06} - 2c^2) X^6 + (Z_{15} - 4c) X^5 Y + \tfrac{9}{4} X^4 Y^2,$$

so that it is possible to reduce either of the first two terms to zero by a proper

choice of the constant c. Let the term in $X^5 Y$ be removed, so that the series will assume the form

$$Z = X^2 + X^3 Y + \Delta_{06} X^6 + \tfrac{9}{4} X^4 Y^2 + \cdots .$$

By applying a transformation of the form (91), viz:

$$X' = \lambda X, \qquad Y' = \mu Y, \qquad Z' = \nu Z,$$

this may be made finally to take the form

$$Z' = X'^2 + X'^3 Y' + kX'^6 + \tfrac{9}{4} X'^4 Y'^2 + \cdots ,$$

where k is unity or zero, according as Δ_{06} does not, or does, vanish. In fact, to accomplish this, it is only necessary to put $\nu = \lambda^2 = \lambda^3 \mu$, and, if $\Delta_{06} \neq 0$, $\lambda = \sqrt[4]{\Delta_{06}}$. If $\Delta_{06} = 0$, λ may be left arbitrary here, and determined subsequently so as to simplify the terms of orders higher than the sixth.

In obtaining the terms of the fifth and sixth orders in this development, we have assumed that the invariant Δ_{13} does not vanish, that is, that the surface is not a cone. If the surface is a cone, Δ_{13} is necessarily zero, and these terms cannot be reduced as in the case just treated. In fact, it can be shown that, in general, the term in X^5 must remain.

The results may be summarized as follows: *For every regular point of a developable surface, a tetrahedron of reference may be so chosen that, in the vicinity of this point, the equation of the surface may be expanded in the form*

$$z = x^2 + k_{13} x^3 y + (1 - k_{13})\, k_{05} (1 - k_{06})\, x^5 + (1 - k_{05})\, k_{06} x^6 + (1 - k_{13})\, k_{15} x^5 y + \tfrac{9}{4} k_{13} x^4 y^2 + R_6, \tag{94}$$

where each of the quantities k_{ij} may have either the value 0 or 1 according as a corresponding invariant Δ_{ij} vanishes or does not vanish, and where R_6 is the remainder after terms of the sixth order.

In reducing the development (85) to the various forms (94), we have purposely avoided the complications which would be involved in preserving the explicit expressions for the coefficients in the several intermediate developments. When these expressions are preserved, the invariants Δ_{ij} appear in the process of reduction. They are

$$\Delta_{13} = \zeta_{13} - 2\zeta_{12}\zeta_{03}, \qquad \Delta_{05} = \zeta_{05} + 2\zeta_{03}^3 - 3\zeta_{03}\zeta_{04},$$

$$\Delta_{15} = \zeta_{15} + 8\zeta_{12}\zeta_{03}^3 - 12\zeta_{12}\zeta_{03}\zeta_{04} + 6\zeta_{12}\Delta_{05},$$

$$\Delta_{13}^4 \Delta_{06} = \Delta_{13}^4 (D_{06} - 4\zeta_{03}\Delta_{05}) - \Delta_{13}^3 \Delta_{05} D_{14} - [D_{14}^2 - 2\Delta_{13}^2 \zeta_{03}^2 - \Delta_{13}(\Delta_{15} - 10\zeta_{12}\Delta_{05} - 4\zeta_{04}\Delta_{13})]^2,$$

where

$$D_{14} = \zeta_{14} + 6\zeta_{12}\zeta_{03}^2 - 3\zeta_{12}\zeta_{04} - 3\zeta_{13}\zeta_{03},$$

$$D_{06} = \zeta_{06} + 3\zeta_{03}^4 - 2\zeta_{03}^2\zeta_{04} - 2\zeta_{04}^2.$$

Consider that form of the development (94) for which Δ_{13} is different from zero, that is,

$$z = x^2 + x^3 y + kx^6 + \tfrac{9}{4}x^4 y^2 + R_6, \tag{94a}$$

where $k = 1$, or 0, according as Δ_{06} is different from, or equal to zero.

Let the transformation of form (88) which will project the surface (94a) into the surface (85) be

$$x = \frac{\lambda_1(\xi + \lambda\zeta)}{1 + \alpha\xi + \beta\eta + \gamma\zeta}, \qquad y = \frac{\mu_1(\rho\xi + \eta + \mu\zeta)}{1 + \alpha\xi + \beta\eta + \gamma\zeta}, \qquad z = \frac{\nu_1\zeta}{1 + \alpha\xi + \beta\eta + \gamma\zeta}.$$

If homogeneous coördinates

$$x_1 = z, \qquad x_2 = y, \qquad x_3 = x, \qquad x_4 = 1, \tag{95}$$

be introduced, the relations between these coördinates and those of the original system are defined by the equations

$$\begin{aligned}
\omega t_1 &= \frac{1}{2\nu_1}x_1, \\
\omega t_2 &= \frac{1}{\nu_1}(\lambda\rho - \mu)x_1 + \frac{1}{\mu_1}x_2 - \frac{\rho}{\lambda_1}x_3, \\
\omega t_3 &= -\frac{\lambda}{\nu}x_1 + \frac{1}{\lambda_1}x_3, \\
\omega t_4 &= \frac{1}{\nu_1}[\alpha\lambda - \beta(\lambda\rho - \mu) - \gamma]x_1 - \frac{\beta}{\mu_1}x_2 + \frac{\beta\rho - \alpha}{\lambda_1}x_3 + x_4,
\end{aligned} \tag{96}$$

where

$$\Delta_{13}\alpha = -2D_{14} - \zeta_{03}\Delta_{13}, \qquad \beta = -\zeta_{12}, \qquad \Delta_{13}\lambda = -D_{14},$$

$$\Delta_{13}^2\rho = -10\lambda^2\Delta_{13} + 2\lambda(5\zeta_{03}\Delta_{13} - 4D_{14}) - 2\Delta_{13}(5\zeta_{03}^2 + 4\zeta_{04}) + 2\zeta_{03}(D_{14} + 3\zeta_{03}\Delta_{13}) - 8\zeta_{12}\Delta_{05} + 2\Delta_{15} - 12\zeta_{12}\Delta_{05},$$

$$\Delta_{13}(\lambda\rho - \mu) + \Delta_{05} = 0, \qquad \gamma = \Delta_{13}\rho + \lambda^2 + \zeta_{03}^2 - \zeta_{04}, \qquad \lambda_1\mu_1 = \Delta_{13}, \qquad \nu_1 = \lambda_1^2$$

and where $\lambda_1 = \sqrt[4]{\Delta_{06}}$ if $\Delta_{06} \neq 0$. If $\Delta_{06} = 0$, λ_1 is to be left arbitrary until the form of the remainder R_6 is determined.

The vertex $(0, 0, 0, 1)$ of the *canonical tetrahedron* is the point y. Let the coördinates which represent the other vertices $(1, 0, 0, 0)$, $(0, 1, 0, 0)$ $(0, 0, 1, 0)$ be denoted by $\mathfrak{q}'$, $\mathfrak{r}'$, and $\mathfrak{s}'$ respectively. *Since* $\mathfrak{r}' = \Omega$. *this vertex of the canonical tetrahedron lies on the cuspidal edge of the integral surface.*

§ 11. *The canonical quartic scroll.*

Let a developable surface S, not a cone, be referred to the canonical tetrahedron (defined by equation (96)), which belongs to one of its points y and consider the quartic surface

$$z = x^2 + x^3 y, \tag{97}$$

which shall be called the *canonical quartic* of the point. The equation of the latter in the homogeneous coördinates (95) is

$$x_1 x_4^3 - x_3^2 x_4^2 - x_3^3 x_2 = 0. \tag{98}$$

The equations of the first, second and third polars of a point (y_1, y_2, y_3, y_4) with respect to the surface are respectively

$$\begin{gathered} y_4^3 x_1 - y_3^3 x_2 - (3y_3^2 y_2 + 2y_4^2 y_3) x_3 + (3y_4^2 y_1 - 2y_4 y_3^2) x_4 = 0, \\ -(6y_3 y_2 + 2y_4^2) x_3^2 + (6y_4 y_1 - 2y_3^2) x_4^2 - 3y_3^2 x_3 x_2 - 4y_4 y_3 x_4 x_3 \\ + 3y_4^2 x_4 x_1 = 0, \\ 3y_1 x_4^3 - 2y_3 x_4^2 x_3 + 3y_4 x_4^2 x_1 - 2y_4 x_4 x_3^2 - 3y_2 x_3^3 - 3y_3 x_3^2 x_2 = 0. \end{gathered} \tag{99}$$

The only singular points of the surface Q, therefore, lie on the line $x_3 = x_4 = 0$, and every point of this line t is a triple point on the surface. Thus the surface is a scroll. The tangent planes to the surface Q along the triple line t are given by the equation

$$y_1 x_4^3 - y_2 x_3^3 = 0. \tag{100}$$

The three tangent planes coincide for $y_1 = 0$, or $y_2 = 0$, so that $\mathfrak{q}' = (1, 0, 0, 0)$ and $\mathfrak{Q} = (0, 1, 0, 0)$ may be called *triple pinch points* on the quartic scroll Q.*

In order to complete the determination of the canonical tetrahedron, we must find out how the scroll Q is determined by the properties of the surface S. It is easily seen that the most general quartic scroll which has the line $x_3 = x_4 = 0$ for a triple line and a pair of triple pinch points on that line, is given by the equation

$$(ax_4 + hx_3)^3 x_1 + (bx_3 + kx_4)^3 x_2 + cx_4^4 + dx_4^3 x_3 + ex_4^2 x_3^2 + fx_4 x_3^3 + gx_3^4 = 0.$$

If this surface passes through the point $y = (0, 0, 0, 1)$, and has the tangent plane τ, or $x_1 = 0$, in common with the surface S, we must have

$$c = k = d = 0.$$

Let the triple line, instead of being $x_3 = x_4 = 0$, be the arbitrary line (not passing through y†)

$$\begin{aligned} x_4 + lx_1 + mx_2 &= 0, \\ x_3 + px_1 + qx_2 &= 0, \end{aligned}$$

and let the pinch points be arbitrary points of this line, viz., the points of intersection of this line with the planes

$$x_1 + \lambda x_2 = 0, \qquad x_2 + \mu x_1 = 0.$$

* This species of scrolls is described by CAYLEY. See *Collected Mathematical Papers*, Vol. VI, p. 312.

† This will be seen to imply only that y is not a point of the cuspidal edge of S.

The equation of the most general quartic scroll having a triple line with two triple pinch points and touching the surface S at the point y will then be obtained by putting,

$$x_1 + \lambda x_2, \quad x_2 + \mu x_1, \quad x_3 + px_1 + qx_2, \quad x_4 + lx_1 + mx_2,$$

in place of x_1, x_2, x_3, x_4, respectively, in

$$(101) \quad \phi(x_1, x_2, x_3, x_4) = (ax_4 + hx_3)^3 x_1 + bx_3^3 x_2 + ex_4^2 x_3^2 + fx_4 x_3^3 + gx_3^4 = 0,$$

that is, in non-homogeneous coördinates

$$(102) \quad \phi(z + \lambda y, \ y + \mu z, \ x + pz + qy, \ 1 + lz + my) = 0.$$

We shall assume that this quartic scroll has contact of the fifth order with S at the point y. Then the development

$$(94a) \quad z = x^2 + x^3 y + kx^6 + \tfrac{9}{4}x^4 y^2 + \cdots,$$

where $k = 0$, or 1, according as Δ_{06} does, or does not, vanish, must satisfy equation (102) up to terms of the fifth order inclusive. The conditions obtained in this way give, if we put $a = 1$,

$$l = q = m = 0, \quad e = b = -1, \quad f = -3p, \quad h = \tfrac{5}{3}p,$$
$$g = -l - \tfrac{10}{3}p^2, \quad \mu = -\tfrac{28}{27}p^3 - pl.$$

Thus, to every regular point y of the surfaces S there belongs a family of ∞^2 quartic scrolls, having contact of the fifth order with S at y, and carrying a pair of triple pinch points on a triple line. The equation of the most general quartic scroll of this kind is

$$(103) \quad [x_4 + lx_1 + h(x_3 + px_1)]^3 x_1 - (x_3 + px_1)^3 (x_2 + \mu x_1) - (x_3 + px_1)^2 (x_4 + lx_1)^2 + f(x_4 + lx_1)(x_3 + px_1)^3 + g(x_3 + px_1)^4 = 0,$$

where f, g, h, μ, are the functions of p and l given above.

Consider a surface Q of this family. Every plane which contains the point y will intersect the surfaces Q and S in plane curves which have contact of at least the fifth order with each other at y. Among these planes there are some which intersect Q and S in a pair of curves having contact of at least the sixth order with each other at y. These planes may be found by substituting the development (94a) for S into equation (103) for the scroll Q, and equating to zero the terms of the sixth order which do not vanish identically. We obtain in this way a sextic homogeneous equation in x and y, which may be reduced to

$$x^4 (4Sx^2 + 4Rxy + 27y^2) = 0,$$

where

$$R = 12l + 13p^2, \qquad S = 3k + 6l^2 + 31p^2 l + 67p^4;$$

or in homogeneous coördinates,

$$x_3^4 (4Sx_3^2 + 4Rx_3 x_2 + 27x_2^2) = 0. \tag{104}$$

This equation determines six tangents of the surface, that is, the generator g counted four times, and two lines t_1, t_2, which we shall call the tangents of sixth order contact between Q and S, because every plane passing through such a line intersects Q and S in a pair of curves having contact of at least the sixth order with each other at y.

The tangent plane τ, or $x_1 = 0$, to the surface S at the point y, intersects the quartic Q in a quartic curve consisting of the generator g counted twice and a conic K, whose equation is

$$gx_3^2 + fx_3 x_4 - x_4^2 - x_3 x_2 = 0.$$

The polar of the point y with respect to the conic K is

$$2x_4 + px_3 = 0.$$

This line intersects K in two points K_1, K_2, whose coördinates are respectively,

$$x_1 = x_3 = 0, \qquad x_1 = 4x_2 - Rx_3 = 0.$$

The tangent t to K at the point K_2 has for its equation

$$4x_2 - Rx_3 = 0,$$

while the tangent of K at K_1 coincides with g.

Consider the four lines g, t, t_1, t_2. They all lie in the plane τ and pass through the point y. The lines t_1 and t_2 will be harmonic conjugates with respect to g and t if, and only if, $R = 0$. But, when $R = 0$, $l = -\frac{13}{12}p^2$, so that there remains a single infinity of quartic scrolls Q_h whose tangents of sixth order contact with S divide g and t harmonically. For every surface of this family the line coincides with the edge $x_1 = x_2 = 0$ of the canonical tetrahedron.

Consider one of the scrolls Q_h of this family. The plane $x_2 + \mu x_1 = 0$, determined by the pinch point $\mathfrak{q}'$ and the line t, intersects Q_h in a quartic curve C_4 whose equations are

$$x_2 + \mu x_1 = 0, \tag{105}$$

$$[x_4 + lx_1 + h(x_3 + px_1)]^3 x_1 - (x_3 + px_1)^2 (x_4 + lx_1)^2 + f(x_4 + lx_1)(x_3 + px_1)^3 + g(x_3 + px_1)^4 = 0,$$

where f, g, h, μ are all constant multiples of positive powers of p. If we refer

the curve C_4 to a set of non-homogeneous coördinates,

$$\rho Z = x, \qquad \rho Y = \mu x_1 + x_2, \qquad \rho X = p x_1 + x_3, \qquad \rho = l x_1 + x_4,$$

its equations become

$$Y = 0, \qquad (1 + hX)^3 Z - X^2 + fX^3 + gX^4 = 0.$$

The necessary and sufficient condition that the left member of the latter equation can be factored into the form

$$(a + bZ + cX)(d + eX)(jX^2 + kXZ + mX + nZ + q),$$

or that the curve degenerates into a pair of lines and a conic, is that $g = 0$. But when $g = 0$, $p = \mu = h = f = 0$, and equation (103) becomes identical with that of the canonical quartic scroll (98).

The canonical quartic scroll Q of a point y is, therefore, completely characterized by the following properties.

1. *It has a triple line with two triple pinch points, $\mathfrak{Q}$, $\mathfrak{q}'$, the first of which lies at the intersection of the cuspidal edge of the given surface S with the generator g passing through the point y.*

2. *It has the contact of the fifth order with S at y.*

3. *The tangent plane τ to S at y intersects Q in the generator counted twice and a conic K tangent to g. The two tangents t_1, t_2, (distinct from g) of sixth order contact between Q and S at y, divide the two tangents g and t, drawn to K from y, harmonically.*

4. *A plane through t and $\mathfrak{q}'$ cuts Q in a curve C_4 which degenerates into a pair of lines and a conic.*

For the canonical quartic scroll, given by equation (98), the point K_2 coincides with the point $\mathfrak{s}'$. Thus all of the vertices, y, $\mathfrak{q}'$, $\mathfrak{r}'$, $\mathfrak{s}'$, of the canonical tetrahedron have now been determined geometrically. In order to complete the geometrical determination of the system of coördinates which leads to the canonical development, it remains to find the unit point of the system. This may be done as follows.

Consider the two planes determined by the pinch point $\mathfrak{q}'$ and the two lines t_1, t_2, whose equations are, by (104),

$$4kx_3^2 + 9x_2^2 = 0.$$

By considering a special case, say that in which $b' = 0$, it can easily be shown that Δ_{06}, and hence k, is, in general, different from zero. We shall treat only that case for which $\Delta_{06} \neq 0$. We may then assume $k = 1$, so that the equations of the two planes are separately

$$2x_3 + 3ix_2 = 0, \qquad 2x_3 - 3ix_2 = 0.$$

They intersect the surface Q in two quartic curves C', C'', whose equations are respectively.

$$x_1 x_4^3 + \tfrac{2}{3} i x_3^4 - x_3^2 x_4^2 = 0, \qquad x_1 x_4^3 - \tfrac{2}{3} i x_3^4 - x_3^2 x_4^2 = 0.$$

The Hessians H', H'', of C' and C'', are respectively

$$x_4^4 (4 i x_3^2 - x_4^2) = 0, \qquad x_4^4 (4 i x_3^2 + x_4^2) = 0.$$

Twenty-two of the twenty-four intersections of the curves C' and C'' coincide with the pinch point $\mathfrak{q}'$. The other two intersections are the points of inflection

$$A(-\tfrac{5}{12}\omega^3, \ \tfrac{2}{3}\omega^2, \ 1, \ 2\omega), \qquad B(\tfrac{5}{12}\omega^3, \ \tfrac{2}{3}\omega^2, \ 1, \ -2\omega),$$

where ω is a fourth root of -1, viz., $\frac{1}{2}\sqrt{2} + i\frac{1}{2}\sqrt{2}$.

The curves C'' and H'' intersect in $\mathfrak{q}'$ counted twenty-two times, and in the points of inflection

$$C(-\tfrac{5}{12}\omega, \ -\tfrac{2}{3}\omega^2, \ 1, \ 2\omega^3), \qquad D(\tfrac{5}{12}\omega, \ -\tfrac{2}{3}\omega^2, \ 1, \ -2\omega^3).$$

The four points A, B, C, D, determine a non-degenerate tetrahedron. If the four fourth roots of -1 are interchanged, these points are permuted according to the four group

$$1, \qquad AB \cdot CD, \qquad BC \cdot DA, \qquad AC \cdot BD.$$

These results may be stated as follows. *The vertices of the canonical tetrahedron are the given point y, the two triple pinch points $\mathfrak{q}'$, $\mathfrak{Q}$ of the canonical quartic scroll Q, and the point of tangency $\mathfrak{s}'$, of a tangent from y to the conic K, which the tangent plane at y to the surface S cuts out of Q. The system of coördinates which gives rise to the canonical development refers to this tetrahedron. If the invariant Δ_{06} does not vanish identically, the unit point is to be taken in such a way that the points of inflection, A, B, C, D, described above, have the coördinates indicated. Since these points may be permuted in four different ways, without altering the system of coördinates, the canonical development may be made in four different ways. For this reason its expression contains the fourth root of a rational invariant.*

When $\Delta_{06} = 0$, $k = 0$; and the lines t_1, t_2 coincide with t. In this case it would be necessary to consider terms of higher order than the sixth in order to determine the unit point of the canonical system of coördinates.

UNIVERSITY OF ILLINOIS, URBANA, ILL.,
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THE SOLUTIONS OF NON-HOMOGENEOUS LINEAR DIFFERENCE EQUATIONS AND THEIR ASYMPTOTIC FORM*

BY

K. P. WILLIAMS

Introduction.

Very important progress has recently been made in the analytic theory of homogeneous linear difference equations.

GALBRUN † has used the Laplace transformation to derive important existence theorems, and has investigated the nature of certain *principal solutions* for large values of the variable.

At about the same time NÖRLUND ‡ applied the theory of factorial series to linear difference equations, showed the existence of the same solutions, and gave their asymptotic form. In addition to the case of polynomial coefficients he has considered the case where the coefficients can be expressed in factorial series. In a more recent paper he has investigated an equation with still more general coefficients.§

By means of a method of successive approximation, and a suitable extension of a contour integral due to GUICHARD, CARMICHAEL || has independently shown the existence of solutions, and has found the bounds of increase and decrease of the solutions in a direction parallel to the real axis.

The latest important contribution to the general theory is by BIRKHOFF.¶ He employs a matrix notation, and shows in a direct manner the existence of certain *intermediate solutions* and of the principal solutions. The asymptotic form of these solutions is determined throughout the complex plane. A modification of the integral used by CARMICHAEL plays an important rôle.

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† Comptes Rendus, vol. 148 (1909), p. 905; vol. 149 (1909), p. 1046; vol. 150 (1910), p. 206; vol. 151 (1910), p. 1114; Acta Mathematica, vol. 36 (1912), pp. 1–68.

‡ Comptes Rendus, vol. 149 (1909), p. 841; *Bidrag til de lineaere Differnsligningers Theori*. Dissertation, Copenhagen, 1910.

§ Mémoires de l'Académie Royale des Sciences et des Lettres de Danemark, Copenhagen; 7th series, vol. 6 (1911), pp. 309–327.

|| These Transactions, vol. 12 (1911), pp. 99–134.

¶ These Transactions, vol. 12 (1911), pp. 243–284.

He finds also the form of the periodic functions which connect the principal solutions, and formulates for the first time the properties which are sufficient to characterize a given system.

While the analytic theory of homogeneous linear difference equations has thus been extensively treated, no general theory has been developed for non-homogeneous equations,* although a number of equations of particular form have been considered (see CARMICHAEL, loc. cit., Introduction).

The purpose of the present paper is to prove the existence of certain *principal solutions* of the system of non-homogeneous equations

$$
\begin{aligned}
g_i(x+1) &= \sum_{j=1}^{n} a_{ij}(x)\, g_j(x) \qquad (i = 1, 2, \cdots, n-1), \\
g_n(x+1) &= \sum_{j=1}^{n} a_{nj}(x)\, g_j(x) + b(x+1),
\end{aligned} \tag{1}
$$

and to develop their asymptotic form. The functions $a_{ij}(x)$ and $b(x)$ will be assumed to be rational,† with poles of order at most μ and ν respectively at infinity, so that we can write

$$
\begin{aligned}
a_{ij}(x) &= x^{\mu}\left(a_{ij} + \frac{a_{ij}^{(1)}}{x} + \cdots\right), \\
b(x) &= x^{\nu}\left(b + \frac{b^{(1)}}{x} + \cdots\right),
\end{aligned} \qquad \text{for } |x| > R.
$$

On account of the gain in simplicity of the formulæ we have taken a system where all but one of the equations are homogeneous. It is, furthermore, evident that the system (1) would be no more general if all the equations were non-homogeneous, since the solution of such a system could be obtained by adding together the solutions of the n systems, of the above type, which would be obtained by striking out the non-homogeneous term from all the equations except the first, second, . . . , last, in turn. Thus the theorems for (1) can be extended readily to a system of n equations, all non-homogeneous.

The method employed is based on a method analogous to that of variation of constants in the theory of linear differential equations. This method has previously been supposed to yield only formal results.‡ It will appear,

* It is possible to reduce a non-homogeneous equation to a homogeneous equation. The application of the general results for a homogeneous equation will show the existence of solutions, but gives no direct means of studying their properties. This method is suggested by WALLENBERG and GULDBERG, *Theorie der linearen Differenzengleichungen*, p. 87.

† Most of the results which will be obtained apply with slight modification to the case where $a_{ij}(x)$ and $b(x)$ have the form of rational functions only at infinity.

‡ WALLENBERG und GULDBERG, p. vii, "... die Methode der Variation der Konstanten, die wegen der mit ihr verbundenen Summationen meist nur formale Bedeutung hat ...."

however, that in general, if not in all cases, the sum formulæ that arise can be interpreted so as to give solutions analytic in the finite plane except for poles.

There is a difference of treatment according as $\mu > 0$, $\mu < 0$, or $\mu = 0$.

In § 2 the reduction of the problem to that of making valid one of four formal types of sums is effected.

In § 3, under the hypothesis that $\mu > 0$, a direct summation to the right is used to derive a *first principal solution* of (1), which we denote by $g_{11}(x)$, $g_{21}(x)$, $\cdots$, $g_{n1}(x)$. This solution is analytic in the finite plane save for poles at determinate points near the negative real axis. In § 4 it is proved that in the right half plane this solution is asymptotic to the series formally satisfying (1).

It is shown in § 5 that the formal solution of (1) obtained by a direct summation to the right, such as is used in § 3, is independent of the fundamental set of solutions of the homogeneous system associated with (1) which is used to determine it. This principle is used in § 6 to determine the asymptotic form of the first solution in the left half plane.

A summation by means of a contour integral to the left is employed in § 7 to determine a *second principal solution*, $g_{12}(x)$, $g_{22}(x)$, $\cdots$, $g_{n2}(x)$, analytic save for poles at determinate points near the positive real axis. This solution is asymptotic to the formal series in the left half plane.

In § 8 the relation between the principal solutions is determined.

It is seen in § 9 that, if $\mu < 0$, the same theory holds except that the rôles of the right and left half planes are interchanged. Similar results are proved in § 10 for the intermediate case $\mu = 0$, except in certain extreme cases.

An application of the general theory is made in § 11 to a single equation of the nth order.

The writer wishes to express his appreciation and thanks to Professor BIRKHOFF for the valuable suggestions and aid that have been received from him.

§ 1. *The fundamental existence theorems for a homogeneous system.*

The system

$$g_i(x+1) = \sum_{j=1}^{n} a_{ij}(x)\, g_j(x) \qquad (i = 1, 2, \cdots, n) \tag{2}$$

is the homogeneous system associated with (1). As we shall make use of its solutions, it will be convenient to enumerate some of their properties. We state below the existence theorems for (2) as they are given by Birkhoff.

There exist in general n sets of series:

$$
\begin{aligned}
s_{1j}(x) &= x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\left(s_{1j} + \frac{s_{1j}^{(1)}}{x} + \cdots\right), \\
s_{2j}(x) &= x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\left(s_{2j} + \frac{s_{2j}^{(1)}}{x} + \cdots\right), \quad (j = 1, 2, \cdots, n), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
s_{nj}(x) &= x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\left(s_{nj} + \frac{s_{nj}^{(1)}}{x} + \cdots\right),
\end{aligned} \tag{3}
$$

each of which will formally satisfy (2), but which in general diverge.

The quantities ρ_j are the roots of the *determinant characteristic equation*

$$|\, a_{ij} - \delta_{ij}\rho \,| = 0, \tag{4}$$

where $\delta_{ij} = 0$ for $i \neq j$, and $\delta_{ii} = 1$, and they are so ordered that

$$|\, \rho_1 \,| \overline{\geq} |\, \rho_2 \,| \overline{\geq} \cdots \overline{\geq} |\, \rho_n \,|. \tag{5}$$

Furthermore, they are all supposed different from zero, that is

$$A = |\, a_{ij} \,| = \rho_1 \rho_2 \cdots \rho_n \neq 0. \tag{6}$$

It is assumed that such series exist, and also that

$$S = |\, s_{ij} \,| \neq 0. \tag{7}$$

We can now state the theorems we shall use.

There are two fundamental sets of solutions of (2), *the first and second principal solutions;** *they may be denoted by* $h_{1j}(x)$, $h_{2j}(x)$, $\cdots$, $h_{nj}(x)$ $(j = 1, 2, \cdots, n)$, *and* $\bar{h}_{1j}(x)$, $\bar{h}_{2j}(x)$, $\cdots$, $\bar{h}_{nj}(x)$ $(j = 1, 2, \cdots, n)$, *respectively. They have the defining property that in any right half plane* $h_{ij}(x) \sim s_{ij}(x)$,† *while in any left half plane* $\bar{h}_{ij}(x) \sim s_{ij}(x)$ $(i, j = 1, 2, \cdots, n)$. *The functions* $h_{ij}(x)$ *are analytic in the finite plane save for poles at the poles of* $a_{ij}(x)$, *and the zeros of the determinant* $|\, a_{ij}(x) \,|$, *or points congruent*‡ *to these points on the left; while the functions* $\bar{h}_{ij}(x)$ *are analytic save for poles at the poles of* $a_{ij}(x)$, *and points congruent on the right.*

While all the results which will be derived could be obtained by making use of only the principal solutions of (2), it will simplify the work to make

* Our first and second principal solutions are the second and first principal solutions, respectively, as given by BIRKHOFF. For the present purpose it is convenient to take them as above.

† This means that for any c the difference between $h_{ij}(x)$ and $k+1$ terms of $s_{ij}(x)$ can be written $x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\zeta_k(x)/x^k$, where $\lim_{x=\infty}\zeta_k(x) = 0$, uniformly, for $-\frac{1}{2}\pi \leqq \arg(x-c) \leqq \frac{1}{2}\pi$ or, in a special case, for $-\frac{1}{2}\pi - \epsilon \leqq \arg(x-c) \leqq \frac{1}{2}\pi - \epsilon$, ϵ arbitrarily small (see GALBRUN, loc. cit.). We say then that $h_{ij}(x)$ is *asymptotic* to $s_{ij}(x)$, or in symbols $h_{ij}(x) \sim s_{ij}(x)$, and call $\zeta_k(x)$ the remainder coefficient.

‡ The points $\cdots x-2, x-1, x, x+1, x+2, \cdots$ are called a congruent set of points.

use of the properties of one of the two sets of intermediate solutions that are given by Birkhoff (loc. cit. § 3).

There exists a set of intermediate solutions $\bar{h}'_{1j}(x), \bar{h}'_{2j}(x), \cdots, \bar{h}'_{nj}(x)$ $(j = 1, 2, \cdots, n)$, *defined only at a sufficient distance from the axis of reals. The elements of these solutions are analytic in the region where they are defined, and are such that* $\bar{h}'_{ij}(x) \sim s_{ij}(x)$ *with regard to* x *in the left half plane,* and with regard to* v $(x = u + v\sqrt{-1})$, *in any right half plane.†*

Although these intermediate solutions are not defined near the real axis, they will be of service to us on account of the fact that they maintain their asymptotic form with regard to v in any right half plane, and because the periodic functions which connect them and the second principal solutions are of a simple nature. The relation between these intermediate and the second principal solutions is given by the formula

$$\bar{h}_{ij}(x) = \varphi_{1j}(x)\bar{h}'_{i1}(x) + \varphi_{2j}(x)\bar{h}'_{i2}(x) + \cdots + \varphi_{j-1,j}(x)\bar{h}'_{i,j-1}(x) + \bar{h}'_{ij}(x) \quad (i, j = 1, 2, \cdots, n), \tag{8}$$

where $\varphi_{1j}(x), \varphi_{2j}(x), \cdots, \varphi_{j-1,j}(x)$ are periodic of period 1, and are analytic at a sufficient distance from the real axis.‡

When we put $j = 1$ in (8) we see that the solution $\bar{h}_{11}(x), \cdots, \bar{h}_{n1}(x)$ (i. e., the one corresponding to the largest $|\rho|$) maintains its asymptotic form in the right half plane with regard to v. These functions form the elements of the first column of the determinant $|\bar{h}_{ij}(x)|$. It is also proved in Birkhoff's paper that $|\bar{h}_{ij}(x)|$, or any minor formed from its first k columns $(k < n)$, is asymptotic to $|s_{ij}(x)|$, or its corresponding minor, with regard to v in the right half plane.

There exists a second set of intermediate solutions $h'_{ij}(x)$, such that $h'_{ij}(x) \sim s_{ij}(x)$ with regard to x in the right and with regard to v in the left half plane.

§ 2. *The formal solution and the operation of summation.*

We are thus assured of the existence of solutions of a homogeneous system. This will enable us to determine easily a formal solution of (1). For this purpose let $l_{1j}(x), l_{2j}(x), \cdots, l_{nj}(x)$ $(j = 1, 2, \cdots, n)$ be any fundamental system of solutions of (2). Write

$$g_i(x) = \sum_{j=1}^{n} \omega_j(x) l_{ij}(x) \qquad (i = 1, 2, \cdots, n), \tag{9}$$

* That is, asymptotic in the sense already defined.

† This means that the difference between $\bar{h}'_{ij}(x)$ and $k+1$ terms of $s_{ij}(x)$ can be written $x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\zeta_k(x)v^{-k}$ where $\lim_{r=\infty}\zeta_k(x) = 0$, $-\frac{1}{2}\pi \leqq \arg(x-c) \leqq \frac{1}{2}\pi$.

‡ Birkhoff, loc. cit., p. 266.

and let us determine the quantities $\omega_1(x), \omega_2(x), \cdots, \omega_n(x)$ so that $g_1(x), g_2(x), \cdots, g_n(x)$ will be a solution of the non-homogeneous system. If we substitute the latter expressions in (1), after first putting

$$\omega_j(x+1) = \omega_j(x) + \Delta\omega_j(x),$$

we find in virtue of (2)

$$\text{(10)} \qquad \begin{aligned} &\sum_{j=1}^{n} \Delta\omega_j(x)\, l_{ij}(x+1) = 0 \qquad (i = 1, 2, \cdots, n-1), \\ &\sum_{j=1}^{n} \Delta\omega_j(x)\, l_{nj}(x+1) = b(x+1). \end{aligned}$$

This is a system of n linear non-homogeneous equations in the n quantities $\Delta\omega_1(x), \Delta\omega_2(x), \cdots, \Delta\omega_n(x)$. Let $L(x)$ represent the determinant $|l_{ij}(x)|$. Further, let $L_j(x)$ be the minor of the element in the last row and jth column of $L(x)$. From (10) we have

$$\text{(11)} \qquad \Delta\omega_j(x) = (-1)^{n+j} \frac{b(x+1)\, L_j(x+1)}{L(x+1)}.$$

We are thus naturally led to the consideration of the simple non-homogeneous equation

$$\text{(12)} \qquad \Delta f(x) \equiv f(x+1) - f(x) = \theta(x).^*$$

There are two series which formally satisfy (12), namely,

$$\text{(13)} \qquad f(x) = -\theta(x) - \theta(x+1) - \cdots,$$

$$\text{(13')} \qquad f(x) = \theta(x-1) + \theta(x-2) + \cdots,$$

which, if they converge, will be actual solutions. These will be called the *direct sums* to the *right* and *left* respectively.

Besides the direct sums, a formal solution of (12) exists in the form of a contour integral due to Guichard.† We shall use it in the form adopted by Carmichael.‡ Let $\theta(x)$ be analytic outside of the region D, extending to the right and lying within a finite distance of the positive half of the real axis (see fig. 1); then if the contour $L = \infty AB\infty$ is drawn so as to enclose D, and passes between $x-1$ and x in the manner shown in the figure, the function

$$\text{(14)} \qquad f(x) = \int_L \frac{\theta(t)\, dt}{e^{2\pi(x-t)\sqrt{-1}} - 1}$$

* The solution written in the form $\sum_x \theta(x)$ is called a sum formula where the operation Σ is such that $\sum_{x+1} \theta(x) - \sum_x \theta(x) = \theta(x)$.

† WALLENBERG und GULDBERG, loc. cit., p. 12.

‡ Loc. cit., p. 119.

formally satisfies (12) by Cauchy's formula, since $f(x+1) - f(x)$ reduces to the integral around $ABB'A'$, and the only singularity of the integrand within this region is at $t = x$.*

If now $\theta(t)$ decreases along $A\infty$ and $B\infty$ in such a way that the integral converges, (14) will furnish an actual solution of (12).

It is obvious that in case the region D extends to the left, we can take a contour L extending to the left so as to again enclose D. The sign of the integral must in this case be changed if we integrate so as to keep the area within L on our left.

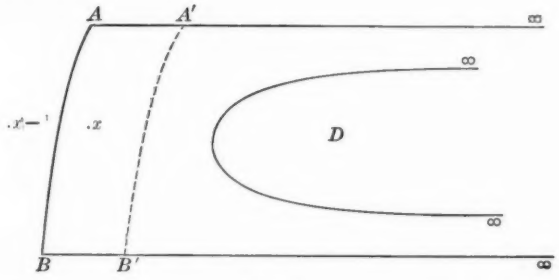

FIG. 1.

We therefore have four formal solutions of an equation of the type (12), the *direct sums* to the *right* and *left* respectively, and *contour integrals* about contours extending to the *right* or *left*.

When one of the above sums is applied to (11) we see that (9) will furnish a formal solution of (1). *Our main problem therefore is to determine whether we can so choose the system of solutions* $l_{1j}(x), l_{2j}(x), \cdots, l_{nj}(x)$ $(j = 1, 2, \cdots, n)$, *of* (2), *that the application of one or more of the four sums to* (11) *will lead to an analytic solution of that equation and therefore of* (1).

§ 3. *The first principal solution for* $\mu > 0$.

We choose first as the solutions of (2) the first principal solutions, $h_{1j}(x)$, $h_{2j}(x), \cdots, h_{nj}(x)$ $(j = 1, 2, \cdots, n)$. Take, further, as the solution of (11) the direct sum to the right. Thus we have formally

$$(15)\quad \omega_j(x) = (-1)^{n+j+1}\left[\frac{b(x+1)H_j(x+1)}{H(x+1)} + \frac{b(x+2)H_j(x+2)}{H(x+2)} + \cdots\right],$$

where $H(x)$ is the determinant $|h_{1j}(x)|$, and $H_j(x)$ is the minor of this with reference to the element in the last row and jth column.

* In case $\theta(x)$ is analytic within D, the integral in (14) reduces to (13) by a familiar theorem in the theory of residues, since the only singularities of the integrand are at $x, x+1, x+2, \cdots$. That is, we are led to (14) on writing (13) as a series of residues.

Each of the terms of (15) will be analytic save at the zeros of $H(x)$, and the poles of $b(x+1)$ and $H_j(x+1)$, or points congruent to these points on the left.

As has been stated, $h_{ij}(x)$ is asymptotic to $s_{ij}(x)$ in the right half plane, so that $H(x)$ is asymptotically represented by the determinant $|s_{ij}(x)|$. From (7) it follows that $H(x)$ will have no zeros in the right half plane at a sufficient distance from the origin. Since the quantities $h_{ij}(x)$ form a system of solutions of the homogeneous system, we see easily that $H(x+1)$ is equal to $A(x)H(x)$, where $A(x)$ is the determinant $|a_{ij}(x)|$. Therefore we have $H(x-1) = H(x)/A(x-1)$, and since $A(x)$ has no singularities in the finite part of the plane exterior to a circle of radius R, it follows that $H(x)$ will not vanish in the finite plane at a sufficient distance from the negative real axis, say at a distance greater than R'.

From the situation of the singularities of $h_{ij}(x)$ it is apparent that $H_j(x)$ is analytic in the finite plane at a distance greater than some quantity $\bar{R}$ from the negative real axis, while $b(x+1)$ is analytic if $x+1$ lies outside a circle of radius R.

Let T be the greatest of the quantities R, R', and $\bar{R}$. About the origin draw a circle of radius T, and the half-lines tangent to it and parallel to the neg-

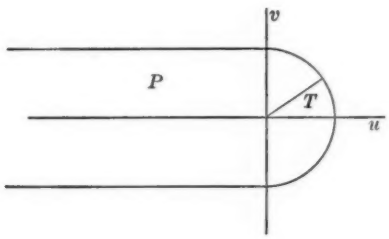

FIG. 2.

ative real axis (fig. 2). Denote by P the region enclosed by these lines and the semi-circumference in the right half plane. Then if x is any point in the finite plane exterior to P, all the terms of (15) are analytic. If x is within the region P, some of the first terms may have singularities, but all the terms from the mth remain analytic, if m be so chosen that $x+m$ is exterior to P.

It remains now to discuss the convergence of (15). Consider the mth term. We will show that it is less in absolute value than the mth term of a rapidly converging series. To do this, replace the elements of $H_j(x+m)$ and $H(x+m)$ by their asymptotic forms. When this is done it is evident that we can remove the factor $(x+m)^{\mu(x+m)+r_k}(\rho_k e^{-\mu})^{x+m}$ from the kth column ($k = 1, 2, \cdots, n$) of $H(x+m)$. All these factors can be removed from $H_j(x+m)$

except the one for $k = j$, so that we have*

$$(16)\quad \frac{b(x+m)H_j(x+m)}{H(x+m)} \sim \frac{(x+m)^{\nu}}{(x+m)^{\mu(x+m)+\tau_j}(\rho_j e^{-\mu})^{x+m}}\left(d_j + \frac{d_j^{(1)}}{x+m} + \cdots\right),$$

where

$$(16')\quad d_j = \frac{bS_j}{S},$$

S_j being the minor of the element in the last row and jth column of $S = |s_{ij}|$.

If we put $x + m = m_1$, we see that the typical term of our series can be written

$$\frac{m_1^c C^{m_1}}{m_1^{\mu m_1}} M(m_1),$$

where c and C are constants, and $M(m_1)$ is a uniformly limited function for m large and x in the vicinity of any point x_1. Since m_1 increases indefinitely in the successive terms of the series, it is evident that the above expression is the general term of a uniformly and rapidly converging series, irrespective of the values of c and C, provided only that $\mu > 0$, which we are assuming for the present.

By Weierstrass's theorem the series (15) will therefore represent an analytic function, provided x is exterior to P; for this makes all the terms analytic. If x is within P, the first terms of (15) may, as seen above, have poles, the remaining part of the series converging uniformly, and accordingly $\omega_j(x)$ may have a pole.

The values of $\omega_1(x), \cdots, \omega_n(x)$ when used in (9) will furnish a solution of (1) which is analytic in the finite part of the plane save for poles within P.

The solution which is obtained in this manner can be extended to the left as follows: let $A_{ji}(x)$ be the minor of the element in the jth row and ith column of $A(x)$, and solve (1) for $g_1(x), \cdots, g_n(x)$, as we may do since $A(x) \not\equiv 0$. This changes the equations to the form

$$(17)\quad g_i(x) = \sum_{j=1}^{n} (-1)^{j+i} \frac{A_{ji}(x)}{A(x)} g_j(x+1) + (-1)^{n+i+1} \frac{A_{ni}(x)}{A(x)} b(x+1) \qquad (i = 1, 2, \cdots, n).$$

The solution can now by (17) be analytically continued to the left.

From this form of (1) we can also determine the location of the singularities of the solution without reference to those of $h_{ij}(x)$. Since the g's are analytic in the finite plane exterior to P, it follows that, in their continuation to the left, singularities will be introduced only within P at the singularities of $b(x+1)$, at the minors of $A(x)$, and at the zeros of $A(x)$. A singularity will in general recur at points congruent on the left.

* The series asymptotically representing the sum, product, or quotient of any number of functions is found by adding, multiplying or dividing their respective asymptotic series according to the ordinary rules for convergent series, BOREL, *Les Séries Divergentes*, p. 30.

By using the first principal solutions of the homogeneous system, and the direct sum to the right, we have thus shown the existence of a first principal solution $g_{11}(x), g_{21}(x), \cdots, g_{n1}(x)$, *of* (1), *which is analytic in the finite plane, save for poles near the negative real axis, congruent on the left to the singularities of* $a_{ij}(x), b(x+1)$, *and the zeros of* $A(x)$.

§ 4. *The asymptotic form of the solution in the right half plane.*

We shall now investigate the asymptotic form of the solution $g_{11}(x)$, $g_{21}(x), \cdots, g_{n1}(x)$. Manifestly for this purpose it is merely necessary to consider more closely the terms of (15). The asymptotic form of the typical term has already been determined, and is given in (16).

Before proceeding it will be convenient to introduce a simplification in notation. When $f(x)$ is asymptotically represented by a power series $s(x)$ of negative integral powers of x, instead of replacing $f(x)$ by the expression which is obtained on breaking off the series $s(x)$ at some point and inserting the remainder coefficient, (see footnote, § 1), we shall put

$$f(x) = [a_0;\ x],$$

giving in each case the constant term a_0 and the argument x in the series.*

The leading factor of (16) can be written

$$\frac{x^{\nu}}{x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j+\mu}} \cdot \frac{\left(1+\frac{m}{x}\right)^{\nu}}{x^{\mu(m-1)}\left(1+\frac{m}{x}\right)^{r_j+\mu(x+m)}(\rho_j e^{-\mu})^m}, \tag{18}$$

the first factor of which is independent of m. We shall omit, for the present, this factor, so that we have to consider the series

$$\sum_{m=1}^{\infty} \frac{(e^{\mu}/\rho_j)^m}{x^{\mu(m-1)}\left(1+\frac{m}{x}\right)^{r_j-\nu+\mu(x+m)}}[d_j;\ x+m]. \tag{19}$$

Consider in the first place the first k terms of (19), $k < |x|$, that is, the finite series

$$\begin{aligned}\frac{e^{\mu}/\rho_j}{\left(1+\frac{1}{x}\right)^{r_j-\nu+\mu(x+1)}}[d_j;\ x+1] + \frac{(e^{\mu}/\rho_j)^2}{x^{\mu}\left(1+\frac{2}{x}\right)^{r_j-\nu+\mu(x+2)}}[d_j;\ x+2] \\ + \cdots + \frac{(e^{\mu}/\rho_j)^k}{x^{\mu(k-1)}\left(1+\frac{k}{x}\right)^{r_j-\nu+\mu(x+k)}}[d_j;\ x+k],\end{aligned} \tag{20}$$

* This notation is similar to that used by Birkhoff, these Transactions, vol. 9 (1908), pp. 373-395.

and let us examine the first term in detail. We have

$$\frac{e^{\mu}/\rho_j}{\left(1+\frac{1}{x}\right)^{r_j-\nu+\mu(x+1)}} = \frac{e^{\mu-\{r_j-\nu+\mu(x+1)\}\log(1+1/x)}}{\rho_j} = \frac{e^{-\frac{2(r_j-\nu)+\mu}{2x}+\cdots}}{\rho_j} = \frac{1}{\rho_j}+\frac{\lambda_1}{x}+\frac{\lambda_2}{x^2}\cdots.$$

If then, in the series $d_j + d_j^{(1)}/(x+1) + \cdots$, we put

$$\frac{1}{x+1} = \frac{1}{x}\left(1-\frac{1}{x}+\cdots\right),$$

the first term of (20) can be written

$$\frac{d_j}{\rho_j}+\frac{\xi_1}{x}+\cdots+\frac{\xi_{\mu k}+E_{\mu k}(x)}{x^{\mu k}},$$

where $E_{\mu k}(x)$ is a uniformly bounded function for $|x|$ sufficiently large and x in the right half plane, since it is composed of remainder terms in uniformly convergent series and the remainder coefficients in asymptotic series.* The terms of (20) from the second on can be treated in the same way; but we notice that their expansions begin with $x^{-\mu}, x^{-2\mu}, \cdots, x^{-(k-1)\mu}$, respectively. The first k terms can therefore be written

$$\frac{d_j}{\rho_j}+\frac{l_1}{x}+\frac{l_2}{x^2}+\cdots+\frac{l_{\mu k}+\eta_{\mu k}(x)}{x^{\mu k}},$$

where $\eta_{\mu k}(x)$ is limited for x large in the right half plane.

It remains now to consider the portion of (19) from the term $m = k+1$ on. We will find an upper limit for the sum of these terms. In the first place we notice that the factor

$$\left(1+\frac{m}{x}\right)^{r_j-\nu+\mu(x+m)},$$

which occurs in the denominator, is greater in absolute value than some quantity K, and in general increases with m, since x is in the right half plane. Also the expression $[d_j; x+m]$ can be replaced by a quantity larger in absolute value. Consequently the portion of our series from the $(k+1)$th term on is less in absolute value than the series whose general term is

$$\frac{M}{K\,|\,x\,|^{\mu(m-1)}}\left|\frac{e^{\mu}}{\rho_j}\right|^m,$$

* To prove that $f(x) \sim a_0 + a_1/x + a_2/x^2 + \cdots$, we need show only that $\zeta_k(x)$ in $f(x) = a_0 + a_1/x + \cdots + [a_k + \zeta_k(x)]/x^k$ remains bounded for every k, since $\zeta_k(x) = [a_{k+1} + \zeta_{k+1}(x)]/x$, and hence $\lim_{x=\infty} \zeta_k(x) = 0$. This principle will be used in what follows.

where M does not increase with $|x|$ and K does not decrease. This is a geometrical progression, whose ratio can be made as small as we please by taking $|x|$ large. The sum of the comparison series is therefore of the order of its first term, which is $1/|x|^{\mu k}$, since we are starting with m equal to $k+1$. This is of the same order as the last term of the expression found above for the first m terms, and consequently (19) can be written $[d_j/\rho_j; x]$, where the series can be taken to as many terms as desired, since k was arbitrarily large.

An expression for $\omega_j(x)$ will be obtained by multiplying $[d_j/\rho_j; x]$ by $(-1)^{n+j+1}$, (see (15)), and by the first factor in (18), which was removed from (19).

The quantity we wish to determine is $\omega_j(x)\, h_{ij}(x)$. Therefore, since $h_{ij}(x) = x^{\mu x}(\rho_j e^{-\mu})^x x^{\nu_j}[s_{ij}; x]$, we have

$$\omega_j(x)\, h_{ij}(x) = x^{\nu-\mu}[(-1)^{n+j+1} d_j s_{ij}/\rho_j; x], \tag{21}$$

Letting $j = 1, 2, \cdots, n$ and adding one finds

$$g_{i1}(x) = x^{\nu-\mu}[h_i; x] \qquad (i = 1, 2, \cdots, n), \tag{22}$$

where

$$h_i = (-1)^n\left(\frac{d_1 s_{i1}}{\rho_1} - \frac{d_2 s_{i2}}{\rho_2} + \cdots + (-1)^{n+1}\frac{d_n s_{in}}{\rho_n}\right).$$

From (6) and (16′) we obtain

$$h_i = \frac{(-1)^n b}{SA}\sum_{j=1}^{n}(-1)^{j+1}\rho_1\rho_2\cdots\rho_{j-1}\rho_{j+1}\cdots\rho_n s_{ij} S_j \tag{23}$$
$$(i = 1, 2, \cdots, n).$$

Let us now try to find series which will formally satisfy (1). Write

$$g_i(x) = x^{\nu-\mu}\left(g_i + \frac{g_i^{(1)}}{x} + \cdots\right) \qquad (i = 1, 2, \cdots, n), \tag{24}$$

and substitute in (1). When we develop the resulting equations according to decreasing powers of x it is evident, on remembering that $\mu > 0$, that we can uniquely determine the successive constants g_i, $g_i^{(1)}$, $\cdots$ by comparing coefficients. The developments in all the equations start with x^ν; equating its coefficients we find

$$0 = \sum_{j=1}^{n} a_{ij} g_j \qquad (i = 1, 2, \cdots, n-1),$$

$$0 = \sum_{j=1}^{n} a_{nj} g_j + b.$$

From this we have for the first coefficient in $g_i(x)$

$$g_i = (-1)^{n+i+1}\frac{bA_{ni}}{A} \qquad (i = 1, 2, \cdots, n), \tag{25}$$

where A_{ni} is the minor of the determinant $A = |\, a_{ij} \,|$ with reference to the element in the last row and ith column.

We call the series $g_1(x), \cdots, g_n(x)$ the formal solution of (1). These series are obtained by direct substitution, and in general diverge.

Since the expansions in the expression for $g_{i1}(x)$ and in the formal series $g_i(x)$ $(i = 1, 2, \cdots, n)$ are of the same form, the question naturally arises as to whether their coefficients are the same. We can show that this is the case in the following way. We know that $g_{11}(x), \cdots, g_{n1}(x)$ as given in (22) satisfy (1), and if we substitute this form of the solutions in (1) and compare coefficients of quantities of the same order we see that we must have precisely the same equations satisfied by $h_i, h_i^{(1)}, \cdots$ that were satisfied by $g_i, g_i^{(1)}, \cdots$ $(i = 1, 2, \cdots, n)$. Since these equations had unique solutions, this means that corresponding coefficients in the series $g_{i1}(x)$ and $g_i(x)$ $(i = 1, 2, \cdots, n)$ are equal.

Thus we see that the *first principal solution is asymptotically represented by the formal solution in the right half plane.*

Although we have shown easily that the constants in the series for $g_{i1}(x)$ must be the same as those in the formal series $g_i(x)$, it is interesting to note that we can show by direct computation that h_i is identical with g_i. To do this we must eliminate the quantities ρ_j and s_{ij} from (23).

When we substitute the formal series (3) in (2) and compare coefficients of $x^{\mu+r_j}$ we find

$$\sum_{k=1}^{n} (a_{ik} - \delta_{ik}\rho_j)\, s_{kj} = 0 \qquad (i = 1, 2, \cdots, n),$$

where $\delta_{ik} = 0$ if $i \neq k$, and $\delta_{ii} = 1$. Since ρ_j is a solution of (4), the last equations have solutions other than $s_{kj} = 0$. The first $n - 1$ of them give

$$\rho_j s_{ij} = \sum_{k=1}^{n} a_{ik} s_{kj} \qquad (i = 1, 2, \cdots, n-1).$$

If now we introduce the quantities $\rho_1, \cdots, \rho_{j-1}, \rho_{j+1}, \cdots, \rho_n$ into the successive columns of the determinant S_{nj} and eliminate the ρ's by means of the above expressions, we obtain the relation

$$\rho_1 \cdots \rho_{j-1}\, \rho_{j+1} \cdots \rho_n\, S_{nj} = A_{n1} S_{1j} + A_{n2} S_{2j} + \cdots + A_{nn} S_{nj}, \tag{26}$$

where A_{ij} and S_{ij} are the minors of A and S respectively with reference to the element in the ith row and jth column, and where we have put $S_{nj} = S_j$, the two being identical by definition.

If in (26) we let $j = 1, 2, \cdots, n$, and substitute in (23), we find for $i = 1$

$$h_1 = \frac{(-1)^n b}{SA} \sum_{k=1}^{n} A_{nk} \sum_{j=1}^{n} (-1)^{j+1} s_{1j} S_{kj} = \frac{(-1)^n b A_{n1}}{A} = g_1,$$

since

$$\sum_{j=1}^{n}(-1)^{j+1}s_{1j}S_{kj}=\begin{cases}S \text{ if } k=1,\\ 0 \text{ if } k\neq 1.\end{cases}$$

In a similar way we can show $h_i = g_i$ $(i = 2, 3, \cdots, n)$.

We summarize the results of this and the preceding section in the following:

THEOREM I. *If $\mu > 0$, the system of non-homogeneous linear difference equations* (1) *has a first principal solution $g_{11}(x), g_{21}(x), \cdots, g_{n1}(x)$. All the constituent elements of this solution are analytic throughout the finite plane, save for poles at the poles of the functions $a_{ij}(x), b(x+1)$, at the zeros of the determinant $|a_{ij}(x)|$, and at points congruent to these points on the left. There exist n series $g_1(x), g_2(x), \cdots, g_n(x)$ which formally satisfy the system* (1), *and whose coefficients can be determined by direct substitution. These series in general diverge, but when x approaches infinity in any right half plane, $g_{i1}(x)$ is represented asymptotically by $g_i(x)$.*

This theorem is supplemented by Theorem II of § 6.

§ 5. *A formal property of the direct sum.*

The solution of (1) which we have found was determined by making use of one of the sets of principal solutions of (2). The question now arises as to whether a different solution of (1) would be obtained if we used a different fundamental system of solutions of (2) and employed again the direct sum to the right. The answer is in the negative as will be shown.

Let $l_{1j}(x), l_{2j}(x), \cdots, l_{nj}(x)$ $(j = 1, 2, \cdots, n)$ be any other fundamental system of solutions of (2). From the theory of homogeneous systems we have

$$\begin{aligned} l_{1j}(x) &= \sum_{i=1}^{n}\varphi_{ij}(x)\,h_{1i}(x),\\ l_{2j}(x) &= \sum_{i=1}^{n}\varphi_{ij}(x)\,h_{2i}(x), \qquad (j=1,2,\cdots,n),\\ &\cdots\cdots\cdots\cdots\\ l_{nj}(x) &= \sum_{i=1}^{n}\varphi_{ij}(x)\,h_{ni}(x), \end{aligned} \tag{27}$$

where the functions $\varphi_{ij}(x)$ are periodic of period 1, and the determinant $\Phi(x) = |\varphi_{ij}(x)| \not\equiv 0$.

Denote by $\bar{g}_i(x)$ $(i = 1, 2, \cdots, n)$ the solution of (1) arising from (27). Then one obtains the formula analogous to (9),

$$\bar{g}_i(x)=\sum_{j=1}^{n}\theta_j(x)\,l_{ij}(x) \qquad (i=1,2,\cdots,n), \tag{28}$$

where the functions $\theta_j(x)$ $(j = 1, 2, \cdots, n)$ are to be determined. As before, let $L(x)$ denote the determinant $|l_{ij}(x)|$, and $L_j(x)$ the minor of this with reference to the element in the last row and jth column; then we have by (15), since the work to that point was purely formal,

$$\theta_j(x) = (-1)^{n+j+1}\left[\frac{b(x+1)L_j(x+1)}{L(x+1)} + \cdots\right] \quad (j = 1, 2, \cdots, n). \tag{29}$$

From (27) it follows that $L(x) = H(x) \cdot \Phi(x)$. Similarly, we can show that

$$L_j(x) = \Phi_{1j}(x)H_1(x) + \Phi_{2j}(x)H_2(x) + \cdots + \Phi_{nj}(x)H_n(x) \quad (j = 1, 2, \cdots, n),$$

where $\Phi_{ij}(x)$ is the minor of $\Phi(x)$ with regard to the element in the ith row and jth column, and $H_j(x)$ is the function in § 3. Therefore, since $|\Phi(x)|$ and all its minors are periodic functions, we obtain

$$\theta_j(x) = \frac{(-1)^{n+j+1}}{\Phi(x)}\sum_{i=1}^{n}\Phi_{ij}(x)\left(\frac{b(x+1)H_i(x+1)}{H(x+1)} + \cdots\right) \quad (j=1, 2, \cdots, n),$$

or by virtue of (15),

$$\theta_j(x) = (-1)^{j-1}\frac{\Phi_{1j}(x)}{\Phi(x)}\omega_1(x) + (-1)^{j-2}\frac{\Phi_{2j}(x)}{\Phi(x)}\omega_2(x) + \cdots + \frac{\Phi_{jj}(x)}{\Phi(x)}\omega_j(x) + \cdots + (-1)^{n-j}\frac{\Phi_{nj}(x)}{\Phi(x)}\omega_n(x).$$

Substitute the values just obtained for $\theta_1(x), \cdots, \theta_n(x)$ in (28); we find

$$\bar{g}_i(x) = \frac{\omega_1(x)}{\Phi(x)}\sum_{k=1}^{n}(-1)^{k+1}\Phi_{1k}(x)l_{ik}(x) - \frac{\omega_2(x)}{\Phi(x)}\sum_{k=1}^{n}(-1)^{k+1}\Phi_{2k}(x)l_{ik}(x) + \cdots + (-1)^{n-1}\frac{\omega_n(x)}{\Phi(x)}\sum_{k=1}^{n}(-1)^{k+1}\Phi_{nk}(x)l_{ik}(x).$$

If we make use now of the values of $l_{ij}(x)$ as given by (27), the last expression becomes, upon rearranging,

$$\bar{g}_i(x) = \frac{\omega_1(x)}{\Phi(x)}\sum_{j=1}^{n}h_{ij}(x)\sum_{k=1}^{n}(-1)^{k+1}\varphi_{jk}(x)\Phi_{1k}(x) - \frac{\omega_2(x)}{\Phi(x)}\sum_{j=1}^{n}h_{ij}(x)\sum_{k=1}^{n}(-1)^{k+1}\varphi_{jk}(x)\Phi_{2k}(x) + \cdots\cdots\cdots\cdots + (-1)^{n-1}\frac{\omega_n(x)}{\Phi(x)}\sum_{j=1}^{n}h_{ij}(x)\sum_{k=1}^{n}(-1)^{k+1}\varphi_{jk}(x)\Phi_{nk}(x).$$

Since

$$\sum_{k=1}^{n}(-1)^{k+1}\varphi_{jk}(x)\Phi_{lk}(x)=\begin{cases}0, & \text{if } j\neq l,\\ (-1)^{l+1}\Phi(x), & \text{if } j=l,\end{cases}$$

the above expression for $\bar{g}_i(x)$ easily simplifies, and finally we have

$$\bar{g}_i(x)=\omega_1(x)\,h_{i1}(x)+\omega_2(x)\,h_{i2}(x)+\cdots+\omega_n(x)\,h_{in}(x).$$

Therefore $\bar{g}_i(x)=g_{i1}(x)$ $(i=1,2,\cdots,n)$, so that we have the same solution obtained in § 3.

This result can be stated as follows: *When the equation* (11) *can be solved by means of a direct sum to the right, the solution of* (1) *obtained by the method of variation of constants is independent of the choice of the fundamental system of solutions of the associated homogeneous system, for in this case the series which occur are convergent, and therefore can be rearranged as above.*

§ 6. *The asymptotic form of the solution in the left half plane.*

By an application of the results of the last section we can determine the asymptotic form of the first principal solution in the left half plane.

Take as the solutions of (2) the intermediate solutions $\bar{h}'_{ij}(x), \bar{h}'_{2j}(x), \cdots, \bar{h}'_{nj}(x)$ $(j=1,2,\cdots,n)$, of § 1. Put

$$g'_{i1}(x)=\sum_{j=1}^{n}\bar{\omega}'_j(x)\,\bar{h}'_{ij}(x) \qquad (i=1,2,\cdots,n), \tag{30}$$

and apply the direct sum to the right in order to determine $\bar{\omega}'_1(x), \cdots, \bar{\omega}'_n(x)$; a formal value of $\bar{\omega}'_j(x)$ will be

$$\bar{\omega}'_j(x)=(-1)^{n+j+1}\left[\frac{b(x+1)\,\bar{H}'_j(x+1)}{\bar{H}'(x+1)}+\cdots\right], \tag{31}$$

where $\bar{H}'(x)=|\bar{h}'_{ij}(x)|$, and $\bar{H}'_j(x)$ is the minor of $\bar{H}'(x)$ with respect to the element in the last row and jth column. The same convergence proof used in § 3 applies here, provided x is sufficiently distant from the real axis, since we have $\bar{h}'_{ij}(x)\sim s_{ij}(x)$ with regard to v in the right half plane.

According to the result of the last section this new solution must be identical with the first principal solution. As derived above it is known to exist only at a sufficient distance from the whole extent of the real axis.

The typical term of (31) will again have the asymptotic form (16), the representation holding with regard to x in the left, and with regard to v in the right half plane. To find the asymptotic form of the series, divide it into two parts: the first k terms, where k is such that $x+k+1=x_1$ is a point in the right half plane near the imaginary axis, and secondly, the remaining terms.

Take first the case where x approaches infinity along any ray making an

acute angle with the negative real axis. The first k terms can again be written in the form (20). By the definition of k we have in this case $k < |x|$, so that the quantity

$$\left(1+\frac{m}{x}\right)^{r_j-\nu+\mu(x+m)} \qquad (m<k)$$

which appears in the denominator of the typical term can be developed in the same way that the first term of (20) was treated, since all the expansions are valid if $m<|x|$. Thus we have immediately that the first k terms of (31) when multiplied by $\bar{h}'_{ij}(x)$ can be put in the form (21).

Consider now the terms of (31) from the $(k+1)$th on. They can be written

$$\sum_{m=0}^{\infty} \frac{b(x_1+m)\,\bar{H}'_j(x_1+m)}{\bar{H}'(x_1+m)}.$$

Since $x_1, x_1+1, \cdots$ are points in the right half plane, the asymptotic form of the typical term of this series is again given by (16), the representation being now with regard to v. The series therefore takes the form

$$\sum_{m=0}^{\infty} \frac{(e^{\mu}/\rho_j)^{x_1+m}}{(x_1+m)^{\mu(x_1+m)+r_j-\nu}} M_m(x_1+m), \tag{32}$$

where $M_m(x_1+m)$ remains uniformly bounded for $|v|$ large.

Let $x_1 = u_1 + v\sqrt{-1}$, $r_j = \xi + \zeta\sqrt{-1}$; then we can so choose x_1 that $\mu u_1 + \xi - \nu > 0$. We can also choose x so that the sign of $\mu v + \zeta$ is the same as that of v. It can then easily be shown that the absolute value of the denominator of the $(m+1)$th term of (32) is greater than $|x_1^{\mu(x_1+m)+r_j-\nu}|$. The absolute value of (32) is therefore less than

$$\frac{M\,|(e^{\mu}/\rho_j)^{x_1}|}{|x_1^{\mu x_1+r_j-\nu}|} \sum_{m=0}^{\infty} \left|\frac{e^{\mu}}{x_1^{\mu}\rho_j}\right|^m,$$

where M is a constant not depending on x. Since $|e^{\mu}/x^{\mu}\rho_j|$ can be made less than unity, the sum of the last series is

$$\frac{M'\,e^{(\mu v+\zeta)\arg x_1}}{|x_1|^{\mu u_1+\xi-\nu}} \left|\left(\frac{e^{\mu}}{\rho_j}\right)^{x_1}\right| < M'\,e^{(\mu v+\zeta)\arg x} \left|\left(\frac{e^{\mu}}{\rho_j}\right)^{x_1}\right|, \tag{33}$$

where M' is bounded.

The terms of (31) from the $(k+1)$th on are also to be multiplied by $\bar{h}'_{ij}(x)$, the absolute value of the principal part of which is $|x|^{\mu u+\xi}\,|(\rho_j e^{-\mu})^x\,|e^{-(\mu v+\zeta)\arg x}$. From (33) the absolute value of the product is seen to be less than $M'\,|x|^{\mu u+\xi}\,|\rho_j e^{-\mu}|^{u-u_1}$. This vanishes more rapidly than any power of $1/x$ along the rays considered, inasmuch as $|\rho_j e^{-\mu}|$ is constant; for along these rays u increases negatively. Consequently the terms

of (31) from the $(k+1)$th do not affect the asymptotic form of $\bar{\omega}_j'(x)\,\bar{h}_{ij}'(x)$, which is therefore given by (21).*

It remains now to consider the case where x approaches infinity parallel to the negative real axis at a sufficient distance from it. A slight modification in the treatment of the first k terms must be made, for the quantity $(1+m/x)^{r_j-\nu+\mu(x+m)}$ cannot, in all the terms, be developed as before, for in the last terms $|m/x|$ will approach very close to unity. The modification to be made is apparent: we divide the first k terms into two parts; the first l of them, where, for instance, l is the smallest integer such that $|x+l|<\frac{1}{2}|x|$ (such a choice is always possible when x goes to infinity parallel to the negative real axis), and secondly the remaining terms.

The first l terms when multiplied by $\bar{h}_{ij}'(x)$ will again have the asymptotic form (21). We can show readily that the second group of terms when multiplied by the same quantity will contribute nothing to the asymptotic from of $\bar{\omega}_j'(x)\,\bar{h}_{ij}'(x)$. The details of the argument are similar to the treatment of the terms after the kth, and we will merely outline them in a note.†

For this case we therefore have again that $\bar{\omega}_j'(x)\,\bar{h}_{ij}'(x)$ is given by (21). From this it follows immediately that $g_{i1}'(x)\sim g_i(x)$ in the left half plane. Remembering that the new solution is identical with the first principal solution, we have:

THEOREM II. *If $\mu>0$, the first principal solution is asymptotic to the formal series in the left half plane at a sufficient distance from the real axis.*‡

* If $f(x)=\varphi(x)+\theta(x)$, where $\varphi(x)\sim x^n(a_0+a_1/x+\cdots)$ along some ray, while $\theta(x)$ becomes small of order $1/a^{|x|}$ $(a>1)$, then $f(x)\sim x^n(a_0+a_1/x+\cdots)$ since $1/a^{|x|}$ is smaller than any power of $1/x$ for x large.

† The terms in question can be written in the form (32), $l\leqq m\leqq k$, and x replacing x_1. We can show first that the absolute value of the typical term would be increased if we changed $(x+m)^{\mu x+r_j-\nu}$ in the denominator to $(x+l)^{\mu x+r_j-\nu}$, and the factor $(x+m)^{\mu m}$ to $x_0^{\mu m}$, where x_0 is the smallest in absolute value of $x+l,\cdots,x+k$. We obtain then for a series greater term by term in absolute value, a geometrical progression, which can be summed, the sum being less than

$$\frac{M|(e^{\mu}/\rho_j)^{x+l}|}{|(x+l)^{\mu x+r_j-\nu}|\cdot|x_0|^{\mu l}}<M|(x/2)^{\nu-\mu x-r_j}(e^{\mu}/\rho_j)^{x}|,$$

since $\left|\dfrac{e^{\mu}}{x_0^{\mu}\rho_j}\right|<1$, and $|x+l|<|x/2|$. The product of this and the absolute value of the leading factor of the asymptotic form of $\bar{h}_{ij}'(x)$ is easily seen to be

$$M|(x/2)^{\nu}2^{\mu x+r_j}|=\frac{M|x|^{\nu}2^{\mu u+\xi}}{2^{\nu}}$$

$(x=u+v\sqrt{-1})$. Since u is practically equal to $-|x|$, and M is a fixed number, this decreases more rapidly than any power of $1/x$.

‡ It should be noted that the asymptotic character of this solution of the non-homogeneous equation is somewhat different from that of the solutions of a homogeneous equation. For instance, the asymptotic form of $\Gamma(x)$ holds only for $-\pi<\arg x<\pi$. That functions

It is interesting here to refer to (17). The quantities $A_{ji}(x)/A(x)$ $(j, i = 1, 2, \cdots, n)$, which occur there have zeros of order μ at infinity. Consequently, if we assume that there exists a constant λ such that $\lim x^{\lambda} g_i(x) = k_i$, k_i finite, as x approaches infinity parallel to the negative real axis, equation (17) gives

$$\lim x^{\lambda} g_i(x) = \sum_{j=1}^{n} (-1)^{j+1} \frac{A_{ji}}{A} \left(1 + \frac{\bar{a}_{ij}}{x} + \cdots\right) \lim \frac{x^{\lambda} g_j(x)}{x^{\mu}}$$

$$+ \lim (-1)^{n+i+1} \frac{bA_{ni}}{A} \left(1 + \frac{\bar{b}_i}{x} + \cdots\right) x^{\lambda+\nu-\mu}.$$

From this relation we see that we must have

$$\lambda = \mu - \nu, \qquad k_i = (-1)^{n+i+1} \frac{bA_{ni}}{A},$$

so that

$$g_{i1}(x) = x^{\nu-\mu} \left[(-1)^{n+i+1} \frac{bA_{ni}}{A} + \zeta_i(x) \right] \quad (i = 1, 2, \cdots, n),$$

where $\zeta_i(x)$ tends to zero as $|x|$ becomes infinite. From (24) and (25) it is clear this is obtained by breaking off the series $g_i(x)$ after the first term.

§ 7. *The second principal solution for* $\mu > 0$.

We shall now show the existence of a solution of (1) defined throughout the finite plane, and whose asymptotic form is known in the whole left half plane. It will of course differ from the first principal solution merely by a solution of the homogeneous system (2).

Take as solutions of (2) the principal solutions on the left, that is, $\bar{h}_{1j}(x)$, $\bar{h}_{2j}(x)$, $\cdots$, $\bar{h}_{nj}(x)$ $(j = 1, 2, \cdots, n)$, where $\bar{h}_{ij}(x) \sim s_{ij}(x)$ in the left half plane. Put

$$g_{i2}(x) = \sum_{j=1}^{n} \bar{\omega}_j(x) \bar{h}_{ij}(x) \qquad (i = 1, 2, \cdots, n). \tag{34}$$

We find in precisely the same way as before

$$\Delta\bar{\omega}_j(x) = (-1)^{n+j} \frac{b(x+1)\bar{H}_j(x+1)}{\bar{H}(x+1)}, \tag{35}$$

where $\bar{H}(x)$ is the determinant $|\bar{h}_{ij}(x)|$, while $\bar{H}_j(x)$ is the minor of this with respect to the element in the last row and jth column.

If we attempt to solve (35) by the direct sum to the left, which we would naturally try in this case, the series obtained is found to diverge. We can,

actually exist with the property specified in Theorem II can be shown by considering the series $1/x + 1/(x+1)^2 + 1/(x+2)^3 + \cdots$. It is easy to prove that this function is asymptotic to its formal expansion in powers of $1/x$, with regard to x, for $-\pi < \arg x < \pi$, and also for $\arg x = \pm\pi$, provided that $|v| > a > 1$, where $x = u + v\sqrt{-1}$.

however, apply the integral formula (14). To prove this, we note first that the situation of the singularities of $\bar{H}_j(x)$, and the fact that $\bar{H}(x) \sim |s_{ij}(x)|$ with regard to v in the right half plane, show that the expression on the right in (35) is analytic outside some region D, such as was described before. Accordingly the expression

$$(36) \qquad \bar{\omega}_j(x) = (-1)^{n+j} \int_L \frac{b(t+1)\bar{H}_j(t+1)}{\bar{H}(t+1)} \frac{dt}{e^{2\pi(x-t)\sqrt{-1}} - 1}$$

will formally satisfy (35), and in case the integral converges suitably it will represent an analytic solution. The contour L is the contour $\infty AB\infty$ used earlier.

To examine the nature of the integrand we make use of the intermediate solutions; for although, as noted above, $\bar{H}(t)$ is asymptotic to the determinant $|s_{ij}(t)|$ with regard to v in the right half plane, the minor $\bar{H}_j(t)$ is not asymptotic to the corresponding minor of $|s_{ij}(t)|$, unless $j = n$, (see § 1). We therefore replace the elements of $\bar{H}_j(x)$ by their values as given by (8). When we do this, the resulting determinant can be simplified so as to give finally

$$(37) \qquad \bar{H}_j(x) = \bar{H}'_j(x) + \varphi_{j+1}(x)\bar{H}'_{j+1}(x) + \cdots + \varphi_n(x)\bar{H}'_n(x),$$

where, as in the preceding section, $H'_j(x)$ is the minor of $\bar{H}'(x) = |\bar{h}'_{ij}(x)|$ with respect to the element in the last row and jth column, while $\varphi_{j+1}(x)$, $\cdots$, $\varphi_n(x)$ are periodic functions, and are analytic at a sufficient distance from the real axis.

The first factor of the integrand in (36) can now be written

$$(38) \qquad \frac{b(t+1)\bar{H}_j(t+1)}{\bar{H}(t+1)} = \sum_{i=j}^{n} \varphi_i(t) \frac{b(t+1)\bar{H}'_i(t+1)}{\bar{H}(t+1)}, \qquad \varphi_j(t) = 1.$$

When we make use of the asymptotic form of the quantities involved, the first term of (38) becomes, (see (16)),

$$(39) \qquad \frac{(e^{\mu}/\rho_j)^{t+1}}{(t+1)^{\mu(t+1)+r_j-\nu}} \left(d_j + \frac{d_j^{(1)}}{t} + \frac{\eta_j(t)}{v} \right),$$

where $\eta_j(t)$ is limited, decreasing for large $|v|$ $(t = u + v\sqrt{-1})$. When $|v|$ is sufficiently great it is evident that (39) will decrease along $A\infty$ and $B\infty$ in approximately the manner of an exponential function, inasmuch as we are assuming $\mu > 0$. Since the functions $\varphi_{j+1}(t)$, $\cdots$, $\varphi_n(t)$ are periodic and limited along any line parallel to the real axis, like results hold for all the terms of (38). The quantity $(e^{2\pi(x-t)\sqrt{-1}} - 1)^{-1}$ remains finite along L; consequently the integral will converge in such a manner that $\bar{\omega}_j(x)$ is an analytic function outside of the region D.

With these values of $\bar{\omega}_j(x)$, equation (34) gives a solution of (1) analytic on the left outside of D. This solution can be extended to the right by means of the equations (1) themselves. From the fact that $a_{ij}(x)$ and $b(x+1)$ are analytic in the finite plane outside a certain circle, the extended solution can have singularities only at points congruent on the right to the singularities of $a_{ij}(x)$ and $b(x+1)$.

We now come to the asymptotic form of the solution. For this purpose we break up the contour L into two contours, L_1 and L_2 in the way that Birkhoff has done.* Here L_1 is a fixed contour $\infty AB\infty$, outside of which the integrand in (36) is analytic, and in case x lies between $A\infty$ and $B\infty$ extended to the left, L_2 is a loop circuit around x and the first point, $x+l$, to the left of AB, that is congruent to x; while if x is above $A\infty$ or below $B\infty$, L_2 is a loop circuit to infinity parallel to the positive real axis. In either case L_1 and L_2 are together equivalent to L, and L_1 can be deformed so that no point congruent to x falls on it.

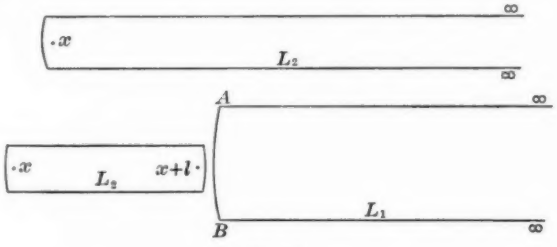

FIG. 3.

Denote the parts of $\bar{\omega}_j(x)$ arising from L_1 and L_2 by $\bar{\omega}_j^{(1)}(x)$ and $\bar{\omega}_j^{(2)}(x)$ respectively. Then

$$g_{i2}(x) = \sum_{j=1}^{n} \bar{\omega}_j^{(1)}(x)\,\bar{h}_{ij}(x) + \sum_{j=1}^{n} \bar{\omega}_j^{(2)}(x)\,\bar{h}_{ij}(x) \quad (i = 1, 2, \cdots, n). \tag{40}$$

Consider first the case where x lies between $A\infty$ and $B\infty$; we have

$$\bar{\omega}_j^{(2)}(x) = (-1)^{n+j+1} \sum_{m=1}^{l+1} \frac{b(x+m)\,\bar{H}_j(x+m)}{\bar{H}(x+m)}. \tag{41}$$

The asymptotic form of the typical term is again given by (16). Also we see

* The reason for this is evident. The singularities of the integrand in (36) are at the congruent points $x, x+1, x+2, \cdots$, due to the zeros of $e^{2\pi(x-t)\sqrt{-1}} - 1$, and the fixed singularities of $\dfrac{b(t+1)\,\bar{H}_j(t+1)}{\bar{H}(t+1)}$, which are in the vicinity of the positive real axis. The residues at the first points will produce a series of the form (13), and the sum of the residues at the last points will be a periodic function. It is therefore natural to consider the two different groups of terms separately.

that precisely the treatment applied to the first k terms of (31) for the case where x approaches infinity parallel to the negative real axis, holds for (41), provided AB is taken far enough to the left. Therefore $\bar{\omega}_j^{(2)}(x)\,\bar{h}_{ij}(x)$ is again given by (21), so that we have

$$\sum_{j=1}^{n} \bar{\omega}_j^{(2)}(x)\,\bar{h}_{ij}(x) \sim g_i(x) \qquad (i=1, 2, \cdots, n), \tag{42}$$

for x between $A\infty$ and $B\infty$.

Next let x lie above $A\infty$ or below $B\infty$; then (41) is replaced by the infinite series

$$\bar{\omega}_j^{(2)}(x) = (-1)^{n+j+1} \sum_{m=1}^{\infty} \frac{b(x+m)\,\bar{H}_j(x+m)}{\bar{H}(x+m)}.$$

Since the fundamental system $\bar{h}_{1j}(x), \bar{h}_{2j}(x), \cdots, \bar{h}_{nj}(x)$ can be expressed in terms of the solutions $h_{1j}(x), h_{2j}(x), \cdots, h_{nj}(x)$ $(j = 1, 2, \cdots, n)$, it follows from the results of § 5 that

$$\sum_{j=1}^{n} \bar{\omega}_j^{(2)}(x)\,\bar{h}_{ij}(x) = \sum_{j=1}^{n} \omega_j(x)\,h_{ij}(x) = g_{i1}(x) \qquad (i = 1, 2, \cdots, n). \tag{43}$$

But we have found that $g_{i1}(x) \sim g_i(x)$ along any ray making an angle with the negative real axis, or parallel to that axis at a sufficient distance from it. We see then that (42) is true throughout the left half plane.

We still have to examine the quantity $\bar{\omega}_j^{(1)}(x)$. L_1 is a fixed contour, so that $\bar{\omega}_j^{(1)}(x)$ is periodic, and, moreover, its absolute value will have a finite upper limit. Furthermore, we can write

$$\bar{\omega}_j^{(1)}(x) = \frac{(-1)^{n+j}}{e^{2\pi x\sqrt{-1}}} \int_{L_1} \frac{b(t+1)\,\bar{H}_j(t+1)}{\bar{H}(t+1)} \frac{dt}{e^{-2\pi t\sqrt{-1}} - e^{-2\pi x\sqrt{-1}}}.$$

The integral approaches a constant, C_j, when x approaches infinity along the negative imaginary axis, so that

$$\lim_{x=\infty} e^{2\pi x\sqrt{-1}}\,\bar{\omega}_j^{(1)}(x) = C_j \qquad \left(\text{for } \arg x = \frac{3\pi}{2}\right). \tag{44}$$

From (36) itself we see

$$\lim_{x=\infty} \bar{\omega}_j^{(1)}(x) = c_j \qquad \left(\text{for } \arg x = \frac{\pi}{2}\right), \tag{45}$$

where c_j is constant.*

Consider now the product $\bar{\omega}_j^{(1)}(x)\,\bar{h}_{ij}(x)$. It can easily be shown that the leading factor $x^{\mu x}(\rho_j e^{-\mu})^x x^{\gamma_j}$ in the asymptotic form of $\bar{h}_{ij}(x)$ vanishes more rapidly than any power of $1/x$ along any ray in the left half plane, including the positive axis of imaginaries. Then, since $\bar{\omega}_j^{(1)}(x)$ remains bounded, $\bar{\omega}_j^{(1)}(x)\,h_{ij}(x)$ will also vanish more rapidly than any power of $1/x$, and con-

*The inversion of order of the double limits employed here is easily justified.

sequently it will not effect the asymptotic form of $\bar{\omega}_j(x)\,\bar{h}_{ij}(x)$, so that

$$g_{i2}(x) \sim g_i(x) \qquad \left(\frac{\pi}{2} \leqq \arg x < \frac{3\pi}{2}\right).$$

By using the second principal solutions of the associated homogeneous system, and expressing the sum formulæ by contour integrals to the right, we have thus proved:

THEOREM III. *If $\mu > 0$, there exists for* (1) *a second principal solution $g_{12}(x)$, $g_{22}(x), \cdots, g_{n2}(x)$ analytic throughout the finite plane, save for poles congruent on the right to the poles of $a_{ij}(x)$ and $b(x+1)$. This solution is such that $g_{i2}(x) \sim g_i(x)$ when x approaches infinity along any ray in the left half plane, inclusive of the positive imaginary axis.*

§ 8. *The relation between the principal solutions.*

The difference between any two solutions of (1) is a solution of the homogeneous system (2). To determine the relation between the principal solutions we make use again of the second principal solutions of the homogeneous system. We know that we must have

$$g_{i2}(x) - g_{i1}(x) = \sum_{j=1}^{n} \theta_j(x)\,\bar{h}_{ij}(x) \qquad (i = 1, 2, \cdots, n),$$

where the functions $\theta_1(x), \cdots, \theta_n(x)$ are periodic. We have really already determined these periodic functions in the form of contour integrals; for a comparison of (40) and (43) with the above expression shows immediately that we must have $\theta_j(x) = \bar{\omega}_j^{(1)}(x)$ when x is sufficiently distant from the real axis.

The functions $\theta_1(x), \cdots, \theta_n(x)$ are therefore analytic at a sufficient distance from the real axis, and as they have no essential singularities in a period strip, they are rational functions of $e^{2\pi x\sqrt{-1}}$.* Thus we can write

$$\theta_j(x) = \sum_{k=1}^{m_j} \sum_{l=1}^{q_k} \frac{D_{kl}^{(j)}}{(2^{2\pi x\sqrt{-1}} - e^{2\pi\alpha_{kj}\sqrt{-1}})^l} + \sum_{k=-n_j}^{\bar{n}_j} d_k^{(j)} e^{2\pi kx\sqrt{-1}} \qquad (j = 1, 2, \cdots, n),$$

where m_j, n_j, $\bar{n}_j$, and q_k are positive integers, and $D_{kl}^{(j)}$, $d_k^{(j)}$ are constants, while α_{kj} is a fixed point.

When we remember the identity of $\theta_j(x)$ and $\bar{\omega}_j^{(1)}(x)$, for x sufficiently distant from the real axis, we see from (44) that we must have

$$d_k^{(j)} = 0, \qquad k \geqq 0$$

$$\sum_{k=1}^{m_j} D_{k1}^{(j)} = C_j. \qquad (j = 1, 2, \cdots, n). \tag{46}$$

* OSGOOD: *Lehrbuch der Funktionentheorie*, I, 1st edition, p. 404, or 2d edition, p. 466.

Likewise from (45) we have

$$d_j^{(k)} = 0, \qquad k < 0,$$

$$\sum_{k=1}^{m_j} \sum_{l=1}^{q_k} \frac{D_{kl}^{(j)}}{(-e^{2\pi a_{kj}\sqrt{-1}})^l} = c_j, \qquad (j = 1, 2, \cdots, n). \tag{47}$$

The relation between the principal solutions is accordingly given by

$$g_{i2}(x) - g_{i1}(x) = \sum_{j=1}^{n} \bar{h}_{ij}(x) \sum_{k=1}^{m_j} \sum_{l=1}^{q_k} \frac{D_{kl}^{(j)}}{(e^{2\pi x\sqrt{-1}} - e^{2\pi a_{kj}\sqrt{-1}})^l} \qquad (i = 1, 2, \cdots, n), \tag{48}$$

with the conditions (46) and (47).

§ 9. *The case* $\mu < 0$.

We have thus far explicitly assumed that μ is positive. We shall now indicate the treatment for μ negative.

The series (15) diverges in this case, and so the direct sum to the right no longer furnishes a solution; but if we apply the direct sum to the left to (35), we can show, by using the asymptotic form of the terms, that the series so obtained will converge uniformly at a sufficient distance from the positive real axis.

By means of a direct sum to the left we thus establish the existence of a solution analytic in the finite plane save for poles at the poles of $a_{ij}(x)$ and $b(x+1)$ and points congruent on the right. This solution corresponds to what was called the second principal solution when μ is positive.

If now we take the first principal solutions as solutions of the homogeneous system, it can easily be shown that the resulting sum formulæ can be taken to be contour integrals about a contour to the left. This gives a second solution of the non-homogeneous system analytic in the finite plane at a sufficient distance from the negative real axis, with its singularities at those of $a_{ij}(x)$, $b(x+1)$, and the zeros of $A(x)$, or points congruent on the left. It thus corresponds to the first principal solution for $\mu > 0$.

These solutions will, as before, be asymptotic in the left and right half planes, respectively, to the series formally satisfying the equations, the first of them maintaining its asymptotic form in the right half plane at a sufficient distance from the positive real axis. The formal series have, however, a somewhat different form, and now are

$$g_i(x) = x^{\nu+\mu}\left(g_i + \frac{g_i^{(1)}}{x} + \cdots\right) \quad (i = 1, 2, \cdots, n-1),$$

$$g_n)(x = x^{\nu}\left(g_n + \frac{g_n^{(1)}}{x} + \cdots\right).$$

For if we substitute expressions of this form in the equations, and compare coefficients of like powers of x, it is evident that at each stage we shall have n equations in which there are n unknown coefficients and coefficients previously determined. Thus the coefficients can be calculated uniquely. In particular we have

$$g_i = a_{in} b \qquad (i = 1, 2, \cdots, n-1),$$

$$g = b.$$

A case of interest, and one which will be used later in the treatment of the equation of the nth order, is that where $a_{ij}(x) = 0$, for $j \neq i+1$, $i \neq n$. In that case the formal series become

$$g_i(x) = x^{\nu+(n-i)\mu}\left(g_i + \frac{g_i^{(1)}}{x} + \cdots\right) \quad (i = 1, 2, \cdots, n-1), \tag{49}$$

where

$$g_i = a_{i,\,i+1}\, a_{i+1,\,i+2} \cdots a_{n-1,\,n}\, b \qquad (i = 1, 2, \cdots, n),$$

$$g_n = b. \tag{50}$$

We can also show the existence of solutions for the case $\mu < 0$ in a second way by referring it to the case $\mu > 0$. When we remember the definition of $A(x)$, $A_{ji}(x)$, the system (17) can be written

$$g_i(x) = \sum_{j=1}^{n} \bar{a}_{ij}(x)\, g_j(x+1) + b_i(x+1) \quad (i = 1, 2, \cdots, n),$$

where

$$\bar{a}_{ij}(x) = x^{-\mu}\left(\bar{a}_{ij} + \frac{\bar{a}_{ij}^{(1)}}{x} + \cdots\right), \qquad \bar{a}_{ij} = (-1)^{j+1}\frac{A_{ji}}{A},$$

$$b_i(x) = x^{\nu-\mu}\left(b_i + \frac{b_i^{(1)}}{x} + \cdots\right), \qquad b_i = (-1)^{n+i+1}\frac{A_{ni}\, b}{A}.$$

If now we put $x = -(z+1)$, $g_i(x) = g_i(-(z+1)) = \bar{g}_i(z+1)$, and $\mu = -\lambda$ $(\lambda > 0)$, this system can be written

$$\bar{g}_i(z+1) = \sum_{j=1}^{n} c_{ij}(z)\, \bar{g}_j(z) + \bar{b}_i(z+1) \quad (i = 1, 2, \cdots, n),$$

where, when $|z| > R$,

$$c_{ij}(z) = z^{\lambda}\left(c_{ij} + \frac{c_{ij}^{(1)}}{z} + \cdots\right), \qquad c_{ij} = (-1)^{j+1}\frac{A_{ji}}{A},$$

$$\bar{b}_i(z) = z^{\nu+\lambda}\left(\bar{b}_i + \frac{b_i^{(1)}}{z} + \cdots\right), \qquad \bar{b}_i = (-1)^{n+i+1}\frac{A_{ni}\,b}{A}.$$

Since λ is positive the previous existence theorems can be applied. The substitution $x = -(z+1)$ shows that the rôles of right and left half planes have been interchanged.

We can summarize the above results as follows:

When $\mu < 0$, first and second principal solutions exist and have their singularities at the same points as before. The first solution is asymptotic to the formal series in the right half plane; the second solution is asymptotic to the same series throughout the left half plane, and also in the right half plane at a sufficient distance from the real axis.

§ 10. *The case* $\mu = 0$.

It still remains to consider the question of solutions for the intermediate case $\mu = 0$. It will appear that at times this case is exceptional.

We shall first examine when the solutions, as found on the hypothesis that $\mu > 0$, are valid when we put $\mu = 0$. Consider the equation (15), and let us see under what conditions the series will still converge. The typical term can now by (16) be written

$$\frac{(x+m)^{\nu-r_j}}{\rho_j^{x+m}}\,[\,d_j;\ x+m\,].$$

If $|\rho_j| > 1$, we shall have a series in which the terms decrease in practically geometrical fashion, so that the series converges uniformly. Since $|\rho_n| \leqq |\rho_i|\,(i < n)$, the series for $\omega_1(x), \cdots, \omega_n(x)$ will all converge if $|\rho_n| > 1$. The first principal solution as found for $\mu > 0$ therefore still exists for $\mu = 0$, if $|\rho_n| > 1$. Under this same condition the second principal solution for $\mu > 0$, given by the contour integrals, is also valid. For in this case (39) can be written

$$\frac{(t+1)^{\nu-r_j}}{\rho_j^{t+1}}\left(d_j + \frac{d_j^{(1)}}{t} + \frac{\eta_j(t)}{v}\right),$$

and this expression decreases along $A\,\infty$ and $B\,\infty$ in such a way that the integral in (36) converges.

We can then say: *If every root of the characteristic equation is greater than unity in absolute value, we can consider the case* $\mu = 0$ *as a limiting case for* $\mu > 0$.

In a similar way we can show that: *When all the roots of the characteristic equation are less than unity in absolute value, the case* $\mu = 0$ *can be considered as a limiting case for* $\mu < 0$.

We still have to determine whether we can make the sum formulæ valid if (1) some of the roots of the characteristic equation are greater in absolute value than unity and others less, or (2) some of the roots of the characteristic equation are equal to unity in absolute value.

Consider the first case. Let $\rho_1, \cdots, \rho_k$ be greater than unity in absolute value, and $\rho_{k+1}, \cdots, \rho_n$ be less. Then, if we take as solutions of the homogeneous equations the first principal solutions, we can use for the sum formulæ for $\omega_1(x), \cdots, \omega_k(x)$ the direct sum to the right. We will next show how to express $\omega_{k+1}(x), \cdots, \omega_n(x)$ as contour integrals. As the work is similar for all of them, we shall consider the equation

$$\Delta\omega_{k+1}(x) = (-1)^{n+k+1}\frac{b(x+1)\,H_{k+1}(x+1)}{H(x+1)}. \tag{51}$$

If x is in the right half plane and is sufficiently large, we know by (16) that the right-hand side of (51) can be written

$$\frac{(x+1)^{v-r_j}}{\rho_{k+1}^{x+1}}[d_{k+1};\ x+1].$$

In § 1 it was stated that there is a set of intermediate solutions of (2) that are asymptotic to the formal series in the right half plane with regard to x, and with regard to v in the left half plane. Suppose these were the solutions of the homogeneous system that we were using; we could then write

$$\Delta\omega_{k+1}(x) = (-1)^{n+k+1}\frac{(x+1)^{v-r_j}}{\rho_{k+1}^{x+1}}[d_{k+1};\ x+1], \tag{51'}$$

where $[d_{k+1};\ x+1]$ is bounded for x large in the right half plane and for v large in the left half plane. Since $|\rho_{k+1}| < 1$, it is evident that for v large enough the above value of $\Delta\omega_{k+1}(x)$ decreases, when x goes to infinity in a direction parallel to the negative real axis, approximately as an exponential. The equation could then be solved by means of a contour integral sum to the left. Return now to the consideration of equation (51). The first principal solutions can be expressed in terms of these intermediate solutions by equations similar to (8), and reference to (37) and (38) shows that the right

hand side of (51) would be replaced by a group of terms such as the one on the right in (51′) in which *every* ρ is less in absolute value than unity. From this we see that $\omega_{k+1}(x)$ can be expressed as a contour integral about a contour to the left, and similarly $\omega_{k+2}(x), \cdots, \omega_n(x)$.

If, therefore, we use the *first principal solutions of the homogeneous system, and divide the resulting sum formulæ into two groups, according as they correspond to roots of the characteristic equation greater in absolute value than unity, or less, we can express the first group by a direct sum to the right, and the second by a contour integral to the left.* Thus we have the *first principal solution* of the non-homogeneous system.

In a similar way we can show that if we use the second principal solutions of the homogeneous system, and divide the sum formulæ as before, *we can solve the first group by a contour integral to the right, and the second by a direct sum to the left.* This gives a *second principal solution.*

The only remaining case is that in which some of the roots of the characteristic equation are equal to unity in absolute value. This case has not been completely investigated for the homogeneous system. Here, in certain cases, at least, a modified sum may give a convergent series. An example of this is the equation occurring in the Weierstrass development of the gamma function, namely the equation

$$g(x+1) - g(x) = \frac{1}{x}.$$

The direct sum to the right is

$$-\left(\frac{1}{x} + \frac{1}{x+1} + \cdots\right),$$

which diverges, while the modified sum

$$\frac{1}{1} - \frac{1}{x} + \left(\frac{1}{2} - \frac{1}{x+1}\right) + \cdots$$

is also a solution, and converges.

When $\mu = 0$ the asymptotic form of the solution requires a slightly different treatment. Take the case where all the roots of the characteristic equation are greater than unity in absolute value, and consider the asymptotic form of the first principal solution.

We need only find the asymptotic form of $\omega_j(x)\, h_{ij}(x)$, where

$$\omega_j(x) = \sum_{m=1}^{\infty} \frac{(x+m)^{\nu-r_j}}{\rho_j^{x+m}} [d_j;\ x+m].$$

Take the first k terms of $\omega_j(x)$, where $\frac{1}{2}|x|-1<k\leqq\frac{1}{2}|x|$; we can write them in the form

$$\frac{x^{\nu-r_j}}{\rho_j^x}\sum_{m=1}^{k}\frac{1}{\rho_j^m}\left(1+\frac{m}{x}\right)^{\nu-r_j}[d_j;\ x+m].$$

In all these terms the expression $(1+m/x)^{\nu-r_j}$ can be developed in a series in $1/x$, since $|m/x|<1$. There is a marked difference, however, between these developments and those for the case $\mu>0$ as given in § 4. In the present case each of the k terms of the summation when expanded will contain a constant term, while in the corresponding expression (20) in the former case, the first term alone when expanded contained a constant term, the others beginning with $x^{-\mu}, \cdots, x^{-(k-1)\mu}$. The final constant term in the present instance will evidently be $d_j\sum_{m=1}^{k}\rho_j^{-m}$. When k increases this approaches $d_j/(\rho_j-1)$, since $|\rho_j|>1$. The first k terms of $\omega_j(x)$, when multiplied by $h_{ij}(x)$, can therefore be written $x^\nu[\lambda_j^{(k)};\ x]$, where $\lambda_j^{(k)}$ approaches $d_j s_{ij}/(\rho_j-1)$ as k increases.

The terms of $\omega_j(x)$ after the kth behave ultimately like a geometrical progression, since the ratio of successive terms obviously approaches $1/\rho_j$ as the number of the terms becomes large. It follows then that the remainder after k terms in the series for $\omega_j(x)$, is, provided k is large enough, of essentially the order of the $(k+1)$th term, which is $M(x+k+1)^{\nu-r_j}/\rho_j^{x+k+1}$, where M is bounded. The product of this and $h_{ij}(x)$ can be written

$$\bar{M}\frac{x^\nu}{\rho_j^{k+1}}\left(1+\frac{k+1}{x}\right)^{\nu-r_j},$$

where $|\bar{M}|$ is less than some fixed constant. Since k increases in a constant ratio to x, this becomes smaller than any power of $1/x$, so that we have ultimately $\omega_j(x)\,h_{ij}(x)=x^\nu[\lambda_j^{(k)};\ x]$ where $\lambda_j^{(k)}$ approaches $d_j s_{ij}/(\rho_j-1)$ for k large.

In this way we can determine the asymptotic form of the solution, and we are led to try to find formal solutions of the form

$$g_i(x)=x^\nu\left(g_i+\frac{g_i^{(1)}}{x}+\cdots\right)\qquad(i=1,\ 2,\ \cdots,\ n).$$

That the coefficients in these expressions can be uniquely determined so as to satisfy the equations can be shown easily by substitution. In particular we find

$$g_i=(-1)^{n+i+1}\frac{b\bar{A}_i}{\bar{A}}\qquad(i=1,\ 2,\ \cdots,\ n),\tag{52}$$

where $\bar{A}$ is the determinant $|a_{ij} - \delta_{ij}|$, ($\delta_{ij} = 0$, if $i \neq j$, $\delta_{ii} = 1$), and $\bar{A}_i$ is the minor of $\bar{A}$ with reference to the element in the last row and ith column. Also we find again that *the first and second principal solutions are asymptotic, in the right and left half planes respectively, to the series formally satisfying the equations.*

§ 11. *The equation of the nth order.*

We shall consider a single non-homogeneous equation of the nth order, which we write in the form

$$g(x+n) + a_1(x)\, g(x+n-1) + \cdots + a_n(x)\, g(x) = b(x). \tag{53}$$

The functions $a_1(x), \cdots, a_n(x), b(x)$ are assumed to be rational, so that we have

$$a_i(x) = x^{i\mu}\left(a_i + \frac{a_i^{(1)}}{x} + \cdots\right), \qquad (\text{for } |x| > R).$$

$$b(x) = x^{\nu}\left(b + \frac{b^{(1)}}{x} + \cdots\right),$$

In place of the single equation (53), consider the system

$$\begin{aligned}
g_1(x+1) &= x^{\mu}\, g_2(x), \\
g_2(x+1) &= x^{\mu}\, g_3(x), \\
&\cdots\cdots\cdots \\
g_{n-1}(x+1) &= x^{\mu}\, g_n(x), \\
g_n(x+1) &= -\sum_{i=1}^{n} \frac{x^{\mu}\left(a_{n-i+1} + \frac{a_{n-i+1}^{(1)}}{x} + \cdots\right)}{\left\{\left(1+\frac{i-1}{x}\right)\left(1+\frac{i}{x}\right)\cdots\left(1+\frac{n-1}{x}\right)\right\}^{\mu}} g_i(x) \\
&\quad + \frac{x^{\nu-(n-1)\mu}\left(b + \frac{b^{(1)}}{x} + \cdots\right)}{\left\{\left(1+\frac{1}{x}\right)\left(1+\frac{2}{x}\right)\cdots\left(1+\frac{n-1}{x}\right)\right\}^{\mu}}.
\end{aligned} \tag{54}$$

If from these equations we eliminate $g_2(x), \cdots, g_n(x)$, we shall find that $g_1(x)$ satisfies (53), so that the system is equivalent to the original equation of the nth order. When we expand the quantities $(1 + j/x)^{\mu}$, which occur in

the last equation above we find that that equation can be written

$$g_n(x+1) = \bar{a}_1(x)\, g_1(x) + \bar{a}_2(x)\, g_2(x) + \cdots + \bar{a}_n(x)\, g_n(x) + b(x+1),$$

where

$$\bar{a}_i(x) = -x^{\mu}\left(a_{n-i+1} + \frac{\bar{a}_i^{(1)}}{x} + \cdots\right),$$

$$\bar{b}(x) = x^{\nu-(n-1)\mu}\left(b + \frac{\bar{b}^{(1)}}{x} + \cdots\right),$$

so that the system is of precisely the form (1). The determinantal characteristic equation reduces to

$$\rho^n + a_1\rho^{n-1} + \cdots + a_{n-1}\rho + a_n = 0, \tag{55}$$

which can therefore be found directly from (53).

We can now apply to the system the existence theorems already established, and we thus have solutions $g_1(x)$ of (53) in all cases except when μ is zero and some of the roots of the characteristic equation are equal to unity in absolute value.

The asymptotic form of the solutions of (54) can be determined by applying the general results for a system (for $\mu < 0$, apply (49)). In each case it is found that the solution $g_1(x)$ is asymptotic to the series formally satisfying the equation (53). This formal solution is now

$$g(x) = x^{\nu-n\mu}\left(g + \frac{g^{(1)}}{x} + \cdots\right) \qquad (\text{for } \mu > 0),$$

$$g(x) = x^{\nu}\left(g + \frac{g^{(1)}}{x} + \cdots\right) \qquad (\text{for } \mu \leqq 0),$$

where

$$g = \frac{b}{a_n} \qquad (\text{for } \mu > 0),$$

$$g = b \qquad (\text{for } \mu < 0)$$

$$g = \frac{b}{1 + a_1 + a_2 + \cdots + a_n} \qquad (\text{for } \mu = 0).$$

The constant term can in all cases be found either from the known form of the constant term of the solution of the corresponding system (for $\mu < 0$, apply (50); for $\mu = 0$, use (52)), or by direct substitution in equation (53).

We can now state:

THEOREM IV. *If $\mu \gtrless 0$, and all the roots of the characteristic equation* (55) *are different from zero, the equation* (53) *has two principal solutions $g_1(x)$ and $g_1'(x)$. The solution $g_1(x)$ is analytic in the finite plane save for possible poles at the poles of $a_1(x), \cdots, a_{n-1}(x), b(x)$, at the zeros of $a_n(x)$, or at points congruent on the left; while $g_1'(x)$ is analytic save for poles at the poles of $a_1(x), \cdots, a_n(x), b(x)$ or points congruent on the right. If $\mu = 0$, the same is true provided in addition that none of the roots of the characteristic equation are equal to unity in absolute value. There exists a series $g(x)$ formally satisfying* (53), *and in all cases in which we have shown the existence of solutions we have $g_1(x) \sim g(x)$ in the right half plane, while $g_1'(x) \sim g(x)$ in the left half plane.*

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AN APPLICATION OF FINITE GEOMETRY TO THE CHARACTERISTIC THEORY OF THE ODD AND EVEN THETA FUNCTIONS*

BY

ARTHUR B. COBLE

Ordinary projective geometry has long been recognized as an important instrument of investigation in other and apparently quite distinct fields, such as the theory of equations and the theory of functions. One of the objects of this paper is to show that a similar purpose is served by the more recently formulated† finite projective geometry. It may be said with some logical justification that this coördination between different subjects is due to their common use of a certain body of abstract theorems. But there are notions, such as that of projection and section, which are so essentially geometric in their origin and significance that their import can only be clouded by viewing them in another or more abstract light.

The characteristic theory of the odd and even theta functions has been the subject of numerous memoirs since the appearance of the original papers of GÖPEL and ROSENHAIN. An excellent account of the present state and a valuable history of the development of this theory is given by KRAZER in his *Lehrbuch Der Thetafunktionen.*‡ As a matter of convenience KRAZER'S notation and formulæ are used in the following. For known theorems also reference is made to Krazer, since often they appear in the original in misleading or distorted form.

The following account is essentially geometrical, the principal notions used being those of linear and quadratic dependence, of the null system, and of projection and section. Practically all of the known theorems are reproved by short and direct methods, which in many cases suggest important generalizations. Two ideas, that of "projection and section of a null system" and

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† VEBLEN and BUSSEY, these Transactions, vol. 7 (1906), pp. 241–59. Much of DICKSON'S *Linear Groups* can be interpreted as finite analytic geometry. Cf. also the author's article, *A Configuration in Finite Geometry*, etc., Johns Hopkins University Circulars, No. 7, 1908; and MITCHELL'S *Determination of the Ordinary and Modular Ternary Linear Groups*, these Transactions, vol. 12 (1911), pp. 207–42.

‡ Leipzig (Teubner), 1903; particularly, pp. 239–304; cited hereafter as K.

that of "section of a null system," are constantly utilized. The grouping of characteristics due to the first of these has been employed to some extent by FROBENIUS; the grouping due to the second seems to have been noted only in a few particular cases.

§ 1 contains an elementary account of the finite geometry (modulo 2) in a linear space in which there may or may not be defined a null system. It is shown in § 2 that the period characteristics behave like the points of an S_{2p-1} with reference to a given null system C. Quadrics in S_{2p-1} are studied in § 3; in particular, those whose polar systems coincide with the null system C. In § 4 the theta characteristics are identified with the quadrics belonging to C. By mapping the quadrics belonging to C upon a space R_{2p}, the period and theta characteristics are shown in § 5 to lie in a linear system. Numerous theorems concerning Steiner and Kummer groups are proved here. § 6 is devoted to the so-called systems of theta characteristics. In the earlier paragraphs translation schemes for the transition from the geometry to the characteristic theory are exhibited.

I think that even a hasty comparison of the presentation here given with the arithmetical method followed by Krazer and others will show that the geometrical point of view is very valuable, not only for suggesting novel ideas but also for giving precision to ideas* with consequent generality of statement.

§ 1. *The Finite Geometry Modulo* 2.

Let $x_0, x_1, \cdots, x_k$ be homogeneous coördinates in the linear space S_k in which the coördinates of points and the coefficients of loci are restricted to the finite number field determined by the modulus 2. The coördinates can take either of the values 0 or 1 but cannot all be zero. If P_k is the number of points in S_k, then

$$P_k = 2^{k+1} - 1, \qquad P_k - P_l = 2^{l+1} P_{k-l-1} \qquad (k > l). \tag{1}$$

An S_{k-1} in S_k is defined by the equation (congruence)

$$u_0 x_0 + u_1 x_1 + \cdots + u_k x_k = 0, \tag{2}$$

or equally well by the coördinates $u_0, u_1, \cdots, u_k$. Thus (2) is the condition that, in S_k, the point x and the S_{k-1} u be incident. In this geometry the fundamental theorems of linear dependence and duality are true; and only those developments which differ essentially from the corresponding developments of ordinary geometry will be considered in detail.

(3) *Given n linearly independent S_{k-1}'s in S_k, the number of points on m of*

* See the closing remarks.

*these S_{k-1}'s and not on the remaining $n-m$ S_{k-1}'s is 2^{k-n+1} if $m<n$, $2^{k-n+1}-1$ if $m=n$.**

Let $z_1=0,\cdots,z_m=0$ be the set of m S_{k-1}'s; and $z_{m+1}=0,\cdots,z_n=0$ be the set of $n-m$ S_{k-1}'s. If $m<n$, the required points are those on the $n-1$ S_{k-1}'s

$$z_1=0,\ \cdots,\ z_m=0,\ z_{m+1}+z_n=0,\ \cdots,\ z_{n-1}+z_n=0,$$

which are not on $z_n=0$ (since $z_{m+1}\equiv 1,\cdots,z_n\equiv 1$ modulo 2). The $n-1$ spaces meet in an S_{k-n+1} which cuts $z_n=0$ in an S_{k-n}. Thus the required number is $P_{k-n+1}-P_{k-n}=2^{k-n+1}$. But if $m=n$ the required points are in an S_{k-n} and are P_{k-n} in number.

The $k+1$ points whose equations are $u_i=0$ are linearly independent and constitute a *point reference basis in* S_k. Also the $k+1$ S_{k-1}'s whose equations are $x_i=0$ constitute an S_{k-1} *reference basis in* S_k. Either basis determines the other and the two constitute a *self-dual reference basis* of S_k. A point reference basis in an S_{k-1} and a point not in the S_{k-1} determine a point reference basis in S_k and each basis is thus determined in $k+1$ ways. If R_k is the number of reference bases in S_k we have, since the number of the S_{k-1} is P_k, the recursion formula

$$R_k=\frac{P_k(P_k-P_{k-1})}{k+1}R_{k-1}=\frac{2^k}{k+1}P_kR_{k-1}.$$

(4) *The number R_k of reference bases in S_k is*

$$R_k=\frac{2^{k(k+1)/2}}{(k+1)!}P_kP_{k-1}\cdots P_1.$$

An S_h in S_k, where $h<k$, is fixed by choosing $h+1$ linearly independent points in S_k; but the same S_h can be thus fixed in R_h ways. The $h+1$ points, when ordered, can be chosen in

$$P_k(P_k-P_0)(P_k-P_1)\cdots(P_k-P_{h-1})$$

ways. This number divided by $R_h\cdot(h+1)!$ is the number of S_h's in S_k. Hence, from (1) and (4),

(5) *For $h<k$, the number $P_k^{(h)}$ of S_h's in S_k is*

$$P_k^{(h)}=\frac{P_kP_{k-1}\cdots P_{k-h}}{P_hP_{h-1}\cdots P_1}.$$

We shall be concerned mainly with an odd space S_{2p-1} and coördinates $x_1,x_2,\cdots,x_p,x_{p+1},x_{p+2},\cdots,x_{2p}$. A set of $2p+1$ points must be linearly related. A set of $2p$ linearly independent points, e. g., $u_1=0,\cdots,u_{2p}=0$,

* K., p. 247, III.

determines uniquely a $(2p+1)$th point, $u_1+u_2+\cdots+u_{2p}=0$, such that any $2p$ of the $2p+1$ points are linearly independent. Call such a set of $2p+1$ points a *point basis in* S_{2p-1}. According to the dual of (3) there is a unique S_{2p-2} which is not on $2p$ points of the basis and therefore must be on the $(2p+1)$th point. If

$$\sigma=\sum_{i=1}^{2p}x_i,$$

the set of $2p+1$ such S_{2p-2}'s derived from the point basis is

$$\sigma-x_1=0,\quad \sigma-x_2=0,\quad \cdots,\quad \sigma-x_{2p}=0,\quad \sigma=0.$$

Evidently any $2p$ of these are linearly independent and the set constitutes *an* S_{2p-2} *basis in* S_{2p-1}. The relation between the point basis and the S_{2p-2} basis is mutual. The basis can be defined similarly in an even dimension, the peculiarity of the odd dimension being that corresponding point and S_{2p-2} are incident. The two bases constitute a *self dual basis* in S_{2p-1}.

(6) *In a self dual basis of* S_{2p-1} *there are* $2p+1$ *incident elements* (*point*, S_{2p-2}). *Any set of* $2p$ *points* [$2p$ S_{2p-2}'s] *of the basis is a point* [S_{2p-2}] *reference basis.*

A point basis is fixed by any one of the $2p+1$ point reference bases in it. Hence, from (4),

(7) *The number of reference bases in* S_{2p-1} *is*

$$N_R=\frac{2^{p(2p-1)}}{(2p)!}P_{2p-1}P_{2p-2}\cdots P_1.$$

The number of bases is

$$N_B=\frac{1}{2p+1}N_R=\frac{2^{p(2p-1)}}{(2p+1)!}P_{2p-1}P_{2p-2}\cdots P_1.$$

A collineation in S_{2p-1} is determined when two ordered point bases are made to correspond; a correlation is determined when an ordered point basis and an ordered S_{2p-2} basis are made to correspond. The totality of collineations or correlations is gotten by fixing one basis and allowing the other to vary, whence the number of each is the number of ordered bases.

(8) *The order of the collineation group in* S_{2p-1} *is*

$$N=(2p+1)!\,N_B=(2p)!\,N_R=2^{p(2p-1)}P_{2p-1}P_{2p-2}\cdots P_1.$$

N *is also the number of correlations in* S_{2p-1} *and* $2N$ *is the order of the correlation group in* S_{2p-1}.

Of particular interest are those correlations for which corresponding point and S_{2p-2} are incident, the so-called *null systems*. If $y_i=0$, $i=1,2,\cdots,2p+1$, is an S_{2p-2} basis,

(9) $C = y_1 y_1' + y_2 y_2' + \cdots + y_{2p+1} y_{2p+1}' = 0, \quad \Sigma y = 0, \quad \Sigma y' = 0,$

is a proper correlation. Since $\Sigma y^2 \equiv (\Sigma y)^2 \pmod 2$, the correlation is a null system. For the ith point of the self dual basis, $y_i = 0$, and $y_k = 1$, $k \neq i$, whence the ith point of the basis corresponds to the ith S_{2p-2} or *the self dual basis y is invariant under C*. Moreover, given any null system C, there are self dual bases invariant under C which are determined as follows. Let z_1 be any point of S_{2p-1} (P_{2p-1} choices), and let w_1 be its null S_{2p-2}. Let z_2 be any point not on w_1 (2^{2p-1} choices), and w_2 be its null S_{2p-2}. On the S_1 $z_1 z_2$, w_1 and w_2 cut out a reference basis and one point of S_1 lies on neither w. Let then z_3 be any point not on w_1 or w_2 and not on $z_1 z_2$ (P_{2p-3} choices) and w_3 be its null S_{2p-2}. The S_2 $z_1 z_2 z_3$ is in $w_1 + w_2 + w_3$ since w_i is on z_i but not z_j and z_k. Thus w_1, w_2, w_3 cut S_2 in a pencil of S_1's and every point of S_2 is on a w. Let then z_4 be a point not on w_1, w_2, w_3 (2^{2p-3} choices), and w_4 be its null S_{2p-2}. The S_3 z_1, z_2, z_3, z_4 is not contained in an S_{2p-2} on the S_{2p-5} w_1, w_2, w_3, w_4, whence the w's cut the S_3 in 4 linearly independent S_2's containing all but one of the points of S_3. Let then z_5 be a point not on $w_1, \cdots, w_4$ and not in the S_3 (P_{2p-5} choices), and w_5 be its null S_{2p-2}. The S_4 $z_1, \cdots, z_5$ is contained in $w_1 + \cdots + w_5$, whence every point of S_4 is on a w.* Let then z_6 be point not on $w_1, \cdots, w_5$ (2^{2p-5} choices), and w_6 be its null S_{2p-2}. Proceeding thus we find, in

$$P_{2p-1} \cdot 2^{2p-1} \cdot P_{2p-3} \cdot 2^{2p-3} \cdot \cdots \cdot P_1 \cdot 2 \cdot P_0$$

ways, a set of $2p+1$ points, $z_1, z_2, \cdots, z_{2p+1}$, and a set of $2p+1$ S_{2p-2}'s, $w_1, w_2, \cdots, w_{2p+1}$, subject only to the relation $\Sigma z = 0$, $\Sigma w = 0$, and such that z_i and w_i are incident while z_i and w_k, $k \neq i$, are not incident. Thus the two sets form a self dual basis which by its mode of formation is invariant under C. Taking account of the order we see that

(10) *The number of self dual bases invariant under the null system C is*

$$N_{BC} = \frac{2p^2}{(2p+1)!} P_{2p-1} P_{2p-3} \cdots P_1.$$

Referred to a basis, C takes the unique form (9). This form is unaltered by the group of order $(2p+1)!$ determined by permutations of the basis, whence the order N_C of the group of C is $(2p+1)! \, N_{BC}$. Since all bases are conjugate under G_N, all proper null systems C are conjugate under G_N and their number is N / N_C.

(11) *All proper null systems are conjugate under G_N. Each is unaltered by a group G_{NC} of order*

$$N_C = 2p^2 P_{2p-1} P_{2p-3} \cdots P_1.$$

* In general, in an S_{2k} $2k+1$ S_{2k-1}'s, whose sum is zero but subject to no other relation, contain all the points of S_{2k}; while in an S_{2k+1}, $2k+2$ similarly related S_{2k}'s contain all but one of the points of S_{2k+1}. Cf. K., pp. 267–9.

The number of proper null systems is

$$\frac{N}{N_C} = 2^{p(p-1)} P_{2p-2} P_{2p-4} \cdots P_2 .$$

If the null S_{2p-2} of x contains x' it contains the line $\overline{xx'}$, and the three null S_{2p-2}'s of points on this line contain the line. Such a line will be called a *null line*, any other line an *ordinary line*.

From the canonical form (9) of C in terms of a self dual basis it is clear that the involution

$$y_1' = y_2, \qquad y_2' = y_1, \qquad y_i' = y_i \quad (i = 3, 4, \cdots, 2p+1)$$

is contained in G_{NC}. Every point of $y_1 + y_2 = 0$, including its null point $1, 1, 0, 0, \cdots, 0$, is a fixed point. Every point not on $y_1 + y_2 = 0$ with coördinates $i, j, k, \cdots (i \neq j)$ is interchanged with $j, i, k, \cdots$, the conjugate pair being on a line with the null point of $y_1 + y_2 = 0$. Since G_{NC} is transitive on the points of S_{2p-1}, it contains a conjugate set of P_{2p-1} such involutions which generate G_{NC}. For the transpositions of the y's generate the subgroup of G_{NC} which leaves a basis unaltered. We have then only to show that one basis $a_1, a_2, \cdots, a_{2p+1}$, self dual under C can be transformed by these involutions into any similar basis, $b_1, b_2, \cdots, b_{2p+1}$. Suppose that $k-1$ points a already coincide with points b. By means of the transpositions we adjust the case so that $a_1 = b_1, \cdots, a_{k-1} = b_{k-1}$ while $a_k \neq b_k$. If $\overline{a_k b_k}$ is an ordinary line the null S_{2p-2}'s of the first $k-1$ points all pass through $a_k + b_k$ and the involution determined by $a_k + b_k$ leaves the first $k-1$ points fixed and transforms a_k into b_k. If $\overline{a_k b_k}$ is a null line, let c_k be a point not on the null S_{2p-2}'s of $a_1, a_2, \cdots, a_k, b_k$. Then the two points $a_k + c_k$, $b_k + c_k$ are on the null S_{2p-2}'s of the first $k-1$ points and the involution of the first followed by that of the second leaves $a_1, a_2, \cdots, a_{k-1}$ unaltered and transforms a_k into b_k. The point c_k is subject to $k+1$ conditions and can always be determined according to (3) until $k = 2p$. But two self dual bases coincide if $2p-1$ of their points respectively coincide. Hence

(12) *The group G_{NC} of the null system C is generated by a conjugate set of P_{2p-1} involutions. Each involution is associated with a point and its null S_{2p-2} in such a way that every point on the S_{2p-2} is fixed and every ordinary line on the point contains a conjugate pair of points.**

§ 2. *Period Characteristics of the Thetas as Points in S_{2p-1} Modulo 2.*

The theta function of p variables $\vartheta(v) = \vartheta(v_1, v_2, \cdots, v_p)$ has $2p$ independent *periods*, each consisting of p quantities $\omega_{1a}, \omega_{2a}, \cdots, \omega_{pa}$, where

* Cf. K, § 6, p. 276.

$\alpha = 1, 2, \cdots, 2p$. That is,

$$\vartheta(v + \omega_\alpha) = \vartheta(v_1 + \omega_{1\alpha}, \cdots, v_p + \omega_{p\alpha}) = E\vartheta(v),$$

E being an exponential factor. The value system

$$\tfrac{1}{2}\sum_{\alpha=1}^{2p} \epsilon_\alpha \omega_{1\alpha}, \cdots, \tfrac{1}{2}\sum_{\alpha=1}^{2p} \epsilon_\alpha \omega_{p\alpha}$$

is called a *half period*, whose *period characteristic* or *Per. Char.* is the set of $2p$ integers $\epsilon_1, \cdots, \epsilon_{2p}$. Two characteristics whose half periods differ by a period are looked upon as not essentially distinct, whence the integers ϵ are reducible (modulo 2). Under integral linear transformation of the periods, the half periods are transformed so that the value of the expression

$$\sum_{\mu=1}^{p} (\epsilon_\mu \eta_{p+\mu} - \epsilon_{p+\mu} \eta_\mu)$$

is unaltered, ϵ and η being any two distinct characteristics.* Naturally the coefficients of the transformation are also reducible modulo 2. The zero characteristic $\epsilon_i = 0$, $i = 1, 2, \cdots, 2p$, differs from the remaining *proper Per. Char.* in that it is unaltered by every transformation. Thus our first fundamental theorem is apparent:

(13) *Under integral linear transformation of the periods of the theta function in p variables, the proper Per. Char. are transformed like the points (or their null S_{2p-2}'s) of a finite space S_{2p-1} modulo 2 under the group G_{NC} of collineations which leaves unaltered the proper null system*

$$C = (x_1 x'_{p+1} - x_{p+1} x'_1) + (x_2 x'_{p+2} - x_{p+2} x'_2) + \cdots + (x_p x'_{2p} - x_{2p} x'_p).$$

Thus properties of sets of Per. Char. which are independent of integral period transformation—and these are the only properties of essential importance—can be inferred from the properties of sets of points in S_{2p-1} with reference to C. The translation proceeds as follows:

Point in S_{2p-1}.	Proper Per. Char.†
Two points on a null line.	Two syzygetic Per. Char.‡
Two points on an ordinary line.	Two azygetic Per. Char.†
(14) Sum of a number of points.	Sum of a number of Per. Char.†
Points of a self dual basis of C.	Fundamental system (F. S.) of $2p + 1$ Per. Char.; such that any two are azygetic.§

* K., pp. 242–3.

† With Per. Char., the term proper will be understood hereafter.

‡ K., p. 244.

§ K., p. 267.

The duality in S_{2p-1} established by C permits of expressing an occurrence in several ways; thus two points are on a null line if either is on the null S_{2p-2} of the other. Since the null lines on a point x are in its null S_{2p-2}, we have

(15) *A given point lies on a null line with $2P_{2p-3}$ other points; on an ordinary line with 2^{2p-1} other points.*

*A given Per. Char. is syzygetic with $2P_{2p-3}$ other Per. Char.; azygetic with 2^{2p-1} other Per. Char.**

The translation of (3) for $k = 2p - 1$ is

(16) *The number of Per. Char. syzygetic with m and azygetic with $n - m$ of n given linearly independent Per. Char. is 2^{2p-n} if $m < n$, and P_{2p-n-1} if $m = n$.*†

From (10) the number‡ of F.S.'s is obtained. Some theorems, such as the first part of the following,§ are self-evident from the geometrical point of view:

(17) *A F.S. of Per. Char. is transformed by integral linear transformation into a F.S. All F.S.'s are conjugate under such transformation.*

Let $x^{(1)}, x^{(2)}, \cdots, x^{(r)}$ be r points of S_{2p-1} which are not linearly related. They lie in an S_{r-1} and form a point reference basis of the S_{r-1}. Any space of dimension equal to or less than $2p - r - 1$ which has no point in common with S_{r-1} will be called a *skew space of S_{r-1} in S_{2p-1}*. The r null S_{2p-2}'s of the points x meet in an S_{2p-r-1} called *the null space of S_{r-1}*. In general S_{2p-r-1} meets S_{r-1} in an S_{m-1} called *the null subspace of S_{r-1}*. The line joining any point of S_{r-1} to any point of its null space, S_{2p-r-1}, is a null line. If the null subspace of S_{r-1} coincides with S_{r-1}, it is called a *null S_{r-1}*. The null space of largest dimension is an S_{p-1} called a *Göpel space*. The translation to Per. Char. is made according to the following table.

(18)

S_{r-1}.	Group E_r of Per. Char. of rank r.‖
Skew space of S_{r-1} in S_{2p-1}.	Group H conjugate to the group E_r.¶
Null subspace of S_{r-1}.	Syzygetic subgroup of E_r.**
Null space of S_{r-1}.	Group adjoint to E_r.‡
Null S_{r-1}.	Syzygetic group E.‡
Göpel space.	Göpel group.‡

A null S_k is determined by any point reference basis in it. The $k + 1$ points can be chosen as follows. Let z_1 be the first point (P_{2p-1} choices), w_1 its null S_{2p-2}. Let z_2 be any point on w_1 other than z_1 ($P_{2p-2} - P_0$ choices), w_2 its null S_{2p-2}. Let z_3 be any point on the S_{2p-3} $w_1\, w_2$ other than a point on the S_1 $z_1\, z_2$ ($P_{2p-3} - P_1$ choices), w_3 its null S_{2p-2}; etc. The null S_k is finally determined by means of an ordered point reference basis in it.

* K., p. 244.
† K., p. 247.
‡ K., p. 268.
§ K., p. 270.
‖ K., p. 291.
¶ K., p. 295.
** K., p. 292.

Since the ordering and the particular basis in S_k are not material we get

$$\frac{1}{(k+1)!}\frac{1}{R_k}P_{2p-1}(P_{2p-2}-P_0)(P_{2p-3}-P_1)\cdots(P_{2p-k-1}-P_{k-1})$$

null S_k's. Or, by making use of (4),

(**19**) *There are**

$$P_{Ck}=\frac{P_{2p-1}P_{2p-3}\cdots P_{2p-2k-1}}{P_1P_2\cdots P_k}$$

null S_k's belonging to C | *syzygetic groups of rank $k+1$.*

Given a space S_{r-1} with the null subspace S_{m-1}, let S_{r-m-1} be a space skew to S_{m-1} in S_{r-1}. Begin the construction of a self dual basis of C by choosing $r-m$ points $y^{(m+1)}, y^{(m+2)}, \cdots, y^{(r)}$ in the S_{r-m-1}. The last point $y^{(r)}$, for example, must be on the $2p-(r-m)$ S_{2p-2}'s which meet in S_{r-m-1} and outside the $r-m-1$ null S_{2p-2}'s of $y^{(m+1)}, \cdots, y^{(r-1)}$. According to (3) such points can be found. Let $x^{(1)}, \cdots, x^{(m)}$ be any point reference basis in S_{m-1}. If the number of points $y^{(i)}$ is odd, the point $\Sigma y^{(i)}$, which is in S_{r-m-1}, is also on the null S_{2p-2} of every point $y^{(i)}$ and of every point $x^{(i)}$ and is therefore in S_{m-1}. But S_{m-1} is skew to S_{r-m-1}. Hence $r-m$ is even.

(**20**) *The difference $r-m$ of dimension of S_{r-1} and its null subspace S_{m-1} is even. A reference basis of S_{r-1}, $x^{(1)}, \cdots, x^{(m)}, y^{(m+1)}, \cdots, y^{(r)}$, can be selected so that every line $\overline{x^{(i)}x^{(k)}}$ and $\overline{x^{(i)}y^{(k)}}$ is a null line while every line $\overline{y^{(i)}y^{(k)}}$ is an ordinary line.*

The difference $r-m$ of rank of E_r and its syzygetic subgroup E_m is even. E_r has reference bases of the form $(\alpha_1), \cdots, (\alpha_m), (\beta_{m+1}), \cdots, (\beta_r)$, where the pairs $(\alpha_i)(\alpha_k)$ and $(\alpha_i)(\beta_k)$ are syzygetic while the $(\beta_i)(\beta_k)$ are azygetic.†

Such a reference basis of S_{r-1} will be called a *normal reference basis.*

Further theorems in this paragraph will be stated in only one form, the translation being obvious.

Since S_{m-1} is part or all of the space common to r S_{2p-2}'s, $m+r \leqq 2p$. Hence a space S'_{m-1} can be found skew to the null space S_{2p-1-m} of S_{m-1} and therefore skew to both S_{m-1} and S_{r-m-1}. Then S'_{m-1} and S_{r-m-1} lie in an S'_{r-1} in which part of a self dual basis of C can be constructed beginning with $y^{(m+1)}, \cdots, y^{(r)}$, in S_{r-m-1}, and ending with $y^{(1)}, \cdots, y^{(m)}$. These m points determine a space S''_{m-1} which may coincide with S'_{m-1}, but which at all events is skew to S_{2p-1-m}, to S_{m-1}, and to S_{r-m-1}. The null S_{2p-2}'s of $y^{(1)}, \cdots, y^{(m)}$ cut S_{m-1} in S_{m-2}'s since these points are not found in S_{2p-1-m}. Thus an S_{m-2} reference basis in S_{m-1} is obtained which carries with it a point reference basis

* For $k=p-1$, see K., p. 296.

† K., p. 294.

$x^{(1)}, \cdots, x^{(m)}$ in S_{m-1}. The $m + r$ points

$$y^{(m+1)}, \cdots, y^{(r)}, y^{(1)}, \cdots, y^{(m)}, z^{(1)} = y^{(1)} + x^{(1)}, \cdots, z^{(m)} = y^{(m)} + x^{(m)}$$

form part of a self dual basis of C. For $y^{(m+i)}$ is azygetic with $y^{(k)}$ by construction and, being syzygetic with $x^{(k)}$, is azygetic with $z^{(k)} = x^{(k)} + y^{(k)}$. Also $y^{(i)}$ is azygetic with $y^{(k)}$ and syzygetic with $x^{(k)}$ by construction hence azygetic with $z^{(k)}$. Also $z^{(i)}$ and $z^{(k)}$ are azygetic since $y^{(i)}$ and $y^{(k)}$ are azygetic while $y^{(i)}$ and $x^{(k)}$, $y^{(k)}$ and $x^{(i)}$, $x^{(i)}$ and $x^{(k)}$, are syzygetic. Since $x^{(i)} = y^{(i)} + z^{(i)}$, we have from (20),

(**21**) *An S_{r-1} with a null subspace S_{m-1} has a reference basis of the form $y^{(1)} + z^{(1)}$, $y^{(2)} + z^{(2)}$, $\cdots$, $y^{(m)} + z^{(m)}$, $y^{(m+1)}$, $\cdots$, $y^{(r)}$, such that the points y, z form part of a self dual basis of C.***

Two self dual bases are conjugate in any order under G_{NC}, whence*

(**22**) *Two spaces of the same dimension in S_{2p-1} are conjugate under G_{NC} if and only if their null subspaces have the same dimension.*

In order to determine the number of S_{r-1}'s in a conjugate set it is convenient to introduce the notions of a "section" and of a "projection and section" of the null system C—notions that can be used later with advantage. According to (20) a space which has no null subspace is of odd dimension, S_{2k-1}. The null S_{2p-2} of a point in S_{2k-1} cannot contain S_{2k-1} else the point is part of a null subspace. Then the null S_{2p-2}'s of the points in S_{2k-1} cut S_{2k-1} in S_{2k-2}'s and thus there is defined a null system C_k in S_{2k-1} which will be called the *section of C by S_{2k-1}*. $2k$ points of a self dual basis of C_k in S_{2k-1}, being azygetic in pairs, are $2k$ points of a self dual basis of C in S_{2p-1} and conversely. The number of such sets in S_{2k-1} is $(2k+1)!\,(N_{BC})_{p=k}$. On the other hand the number of such sets in S_{2p-1} [see the enumeration before (10)] is

$$\frac{1}{(2k)!}\,2^{2p-1}\cdot 2^{2p-3}\cdot\ldots\cdot 2^{2p-2k+1}\,P_{2p-1}\,P_{2p-3}\cdots P_{2p-2k+1}.$$

Dividing this number by the first we find that

(**23**) *The number of S_{2k-1}'s without a null subspace is*

$$2^{2k(p-k)}\,\frac{P_{2p-1}\,P_{2p-3}\cdots P_{2p-2k+1}}{P_{2k-1}\,P_{2k-3}\cdots P_1}.$$

Each is unaltered by a subgroup of G_{NC} of order

$$2^{p^2-2pk+2k^2}\,P_{2p-2k-1}\,P_{2p-2k-3}\cdots P_1\cdot P_{2k-1}\cdots P_3\,P_1.$$

Let S_{r-1} have the null subspace S_{m-1}. The null space of S_{m-1} is S_{2p-m-1} which contains S_{m-1}. If $\pi = p - m$, there are within S_{2p-m-1} and on S_{m-1} precisely $P_{2\pi-1}$ spaces S_m. These we regard as "points Σ_0 in a space $\Sigma_{2\pi-1}$

* K., p. 295.

or Σ." More generally then, spaces S_{r-1} within S_{2p-m-1} and on S_{m-1} are "spaces Σ_{r-m-1} in Σ." The null space of an S_m within S_{2p-m-1} and on S_{m-1} is an S_{2p-m-2} within S_{2p-m-1} and on S_m and therefore on S_{m-1} also or "a point Σ_0 is on its null $\Sigma_{2\pi-2}$ with reference to the thus defined null system Γ_π in Σ." This null system Γ_π in the derived space $\Sigma_{2\pi-1}$ we call the *projection and section of C in S_{2p-1} from the null space S_{m-1} and by the null space S_{2p-m-1} of S_{m-1}*. Two points of Σ are syzygetic or azygetic according as the null space of one corresponding S_m does or does not contain the other corresponding S_m. A space S_{s-1} within S_{2p-m-1} and on S_{m-1} has a null subspace $S_{m'-1}$ which contains S_{m-1}. To S_{s-1} there corresponds in Σ a space S_{s-m-1} which has with reference to Γ_π a null subspace $\Sigma_{m'-m-1}$ and conversely. In particular for $s = r$ and $m' = m$ we see that an S_{r-1} with the null subspace S_{m-1} corresponds to a Σ_{r-m-1} in Σ without a null subspace with reference to Γ_π. The number of these Σ_{r-m-1}'s has been determined in (23) where $2k$ is to be replaced by $r - m$ and p by $\pi = p - m$. This is the number of S_{r-1}'s with a *given* null S_{m-1} as null subspace. The number of null S_{m-1}'s is furnished by (19), whence

(**24**) *The number of S_{r-1}'s with a null subspace S_{m-1}, where $r - m$ is even, is*

$$2^{\frac{r-m}{2}(2p-r-m)} \frac{P_{2p-1} P_{2p-3} \cdots P_{2p-m-r+1}}{P_{r-m-1} P_{r-m-3} \cdots P_1 \cdot P_{m-1} P_{m-2} \cdots P_2 P_1}.$$

They all are conjugate and each is unaltered by a subgroup of G_{NC} of order

$$2^{p^2-p(r-m)+\frac{1}{2}(r^2-m^2)} P_{2p-m-r-1} P_{2p-m-r-3} \cdots P_1 \cdot P_{r-m-1} P_{r-m-3} \cdots P_1 \cdot P_{m-1} P_{m-2} \cdots P_1.$$

§ 3. *Quadrics in S_{2p-1} Modulo 2.*

A *quadric* in S_{2p-1} is defined by a congruence or equation of the form

$$f(xx) = \Sigma a_{ik} x_i x_k = 0 \quad (i, k = 1, \cdots, 2p; i \leqq k). \tag{25}$$

Points whose coördinates do or do not satisfy this equation will be called *quadric* or *outside points* respectively. The quadric will be called *proper* or *degenerate* according as it cannot or can be transformed by a collineation of G_N into a form in less than $2p$ variables.

The point $x + y$ is on the quadric if

$$f(xx) + f'(xy) + f(yy) = 0, \tag{26}$$

where

$$f'(xy) = \Sigma a_{ik} (x_i y_k + x_k y_i) \quad (i, k = 1, \cdots, 2p; i < k).$$

Since

$$x_i y_k + x_k y_i \equiv x_i y_k - x_k y_i = \pi_{ik},$$

we see* that the polar system of $f, f'(xy) = 0$, is a null system whatever be the values of the coefficients a_{ii}.

(27) *The 2^{2p} quadrics f, obtained by varying the coefficients a_{ii}, have as polar systems the same null system. Conversely, any null system determines 2^{2p} quadrics whose polar systems coincide with the given null system.*

An important relation between the quadric and its polar system is this:

(28) *A quadric is proper or degenerate according as its polar system is a proper or degenerate null system.*

For if the quadric is degenerate it can be transformed into a quadric in $2p-1$ variables at most, say $y_2, \cdots, y_{2p}$, and the polar of $1, 0, \cdots, 0$ is indeterminate, i. e., the null system is degenerate. Conversely, if the polar of $1, 0, \cdots, 0$ is indeterminate the quadric either has the form $g(y_2, \cdots, y_{2p})$, and is degenerate or has the form $y_1^2 + g(y_2, \cdots, y_{2p})$. In the latter case, $g(y_2, \cdots, y_{2p})$ is a quadric in an even space whose polar system is a null system and therefore necessarily degenerate. Let $y_2 = 1$, $y_{2+k} = 0$, be a singular point. Then g has either the form $y_2^2 + h(y_3, \cdots, y_{2p})$ or the form $h(y_3, \cdots, y_{2p})$. The original quadric has either the form $z^2 + h(y_3, \cdots, y_{2p})$, where $y_1^2 + y_2^2 \equiv (y_1 + y_2)^2 = z^2$ or the form $y_1^2 + h(y_3, \cdots, y_{2p})$, and in either case is degenerate.

We shall say that a *quadric belongs to* C if its polar system is the null system C. A line will be called a *skew line, tangent, secant,* or *generator of the quadric,* according as it has $0, 1, 2,$ or 3 points in common with the quadric. It is easily verified from (26) that

(29) *The 2^{2p} quadrics which belong to a given null system C have null lines for tangents and generators, ordinary lines for secants and skew lines.*

Thus if x is a point of the quadric, u its null, or polar, or tangent S_{2p-2}, all lines on x in u are generators or tangents of the quadric, and all lines on x and not in u are secants of the quadric; if however x is an outside point, all lines on x in u are tangents of the quadric, and all lines on x and not in u are secants or skew lines of the quadric.

(30) *Quadrics in S_k have real points if $k > 1$.*

For, if (25) contains no reference point, every $a_{ii} \neq 0$; if no point like $1, 1, 0, \cdots, 0$, every $a_{ik} \neq 0$, but then it must contain points like $1, 1, 1, 0, \cdots, 0$.

Let $C(x, x') = 0$ be the equation of the null system C, $f(x, x) = 0$ be a quadric belonging to C, y be a point on f, and I_y be the involution determined by y [see (12)]. $C(x, y) = 0$ is the tangent space of f at y and contains the tangents and generators of f which pass through y. I_y transforms f into a quadric $\varphi(x, x) = 0$, which has in common with f the point y and its tangent space. Any line through y and not in the tangent space has a further point

* Cf., DICKSON, *Linear Groups*, p. 201, footnote.

on f and a further distinct point on φ. Thus f, φ, and $[C(x, y)]^2$ constitute a pencil, or since

$$[C(x, y)]^2 \equiv C(x^2, y^2) \equiv C(x^2, y) \pmod 2,$$

we have

$$f(x, x) + C(x^2, y) = \varphi(x, x).$$

If z is another point on f,

$$f(x, x) + C(x^2, z) = \psi(x, x),$$

whence

$$\varphi(x, x) + \psi(x, x) = C(x^2, y + z).$$

If $\overline{yz}$ is a null line, the point $y + z$ is on f, φ, ψ; if $\overline{yz}$ is an ordinary line, $y + z$ is not on f nor $C(x^2, y) \equiv C(x, y)$ and therefore is on φ and similarly is on ψ. Then φ and ψ also are in a pencil with the square of their common tangent space, $C(x^2, y + z)$. If f contains n points, the $n + 1$ quadrics, f, φ, ψ, being conjugate with f each contain n points and each determines the set by means of its tangent spaces. If r is a point outside f, I_r leaves unaltered the contacts of tangents from r to f but interchanges the points on a secant line through r, i. e., leaves f unaltered. Hence the set of $n + 1$ quadrics is a complete conjugate set under G_{NC}.

Let y and z be two points not on f, i. e., let $f(y, y) = 1$ and $f(z, z) = 1$. Let

$$f(x, x) + C(x^2, y) = \varphi(x, x), \qquad f(x, x) + C(x^2, z) = \psi(x, x).$$

Then

$$\varphi(x, x) + \psi(x, x) = C(x^2, y + z).$$

But $y + z$ is a point of φ and ψ, since

$$\varphi(y + z, y + z) = f(yy) + C(y, z) + f(zz) + C(z, y) \equiv 0.$$

Hence the set of quadrics obtained by adding to f the squares of its secant spaces is such that any one of the set differs from the others by the squares of its own tangent spaces. Again this set is a complete conjugate set. But all the quadrics which belong to C differ from any one by the square of a tangent or secant space of the one. Hence there are only two distinct types of quadrics which belong to C. Since any proper quadric belongs to a proper null system and all proper null systems are conjugate under G_N there are only two distinct types of proper quadrics in S_{2p-1}.*

G_{NC} is doubly transitive on either conjugate set of quadrics. For two quadrics in the set conjugate to f are associated with points y and z on f. If $\overline{yz}$ is an ordinary line, I_{y+z} leaves f unaltered and interchanges y and z. If yz

* Cf. DICKSON, *Linear Groups*, p. 197.

is a null line there is a point t on f and not on the tangent spaces of y and z; e. g., an ordinary line on y and a point outside the tangent space of z meets the quadric again at a point t. The product $I_{y+t}\, I_{t+z}$ leaves f unaltered and sends y into z.

G_{NC} is simultaneously simply transitive on both sets of quadrics. For if y and z are two points not on f and if $\overline{yz}$ is an ordinary line, then $y+z$ is not on f and I_{y+z} leaves f unaltered and interchanges y and z. If $\overline{yz}$ is a null line touching f at $y+z$, let t be a point on the tangent space of $y+z$ but not on f nor $C(x, y)$ and therefore not on $C(x, z)$. Then $\overline{yt}$ and $\overline{zt}$ are skew to f and the product $I_{y+t}\, I_{z+t}$ leaves f unaltered and sends y into z. But y and z determine any two quadrics of the set which does not contain f.

(31) *The 2^{2p} proper quadrics which belong to C divide into two sets conjugate under G_{NC}. Quadrics from the same set have contact along an S_{2p-2} tangent to both; quadrics from different sets have contact along a space secant to both. Every proper quadric in S_{2p-1} is conjuagte under G_N with one or the other type. G_{NC} is doubly transitive on the quadrics in either set; and simultaneously simply transitive on the quadrics of both sets. The group of any quadric is simply transitive on the points of the quadric and also on the outside points of the quadric.*

Let there be τ points on, and $(P_{2p-1}-\tau)$ points outside, the proper quadric f which belongs to C. According to (31) the quadric points are each of the same type; the same is true of the outside points. Let there be on an outside point, ρ_0, ρ_1, ρ_2 lines skew, tangent, secant, respectively to f; on a quadric point, σ_1, σ_2, σ_3 lines respectively tangent to, secant to, on, f. By joining an outside point to the other outside points and to the quadric points we get the equations

$$2\rho_0+\rho_1=P_{2p-1}-\tau-1, \qquad \rho_1+2\rho_2=\tau.$$

In the same way from a quadric point we get

$$2\sigma_1+\sigma_2=P_{2p-1}-\tau, \qquad \sigma_2+2\sigma_3=\tau-1.$$

But $\rho_1=\sigma_1+\sigma_3$ is the number of null lines on a point, i. e., P_{2p-3}. Hence we have, in terms of τ,

$$2\rho_0=P_{2p-1}-P_{2p-3}-\tau-1, \qquad \rho_1=P_{2p-3}, \qquad 2\rho_2=\tau-P_{2p-3},$$

$$2\sigma_1=P_{2p-1}-\tau-2^{2p-2}, \qquad \sigma_2=2^{2p-2}, \qquad 2\sigma_3=\tau-1-2^{2p-2}.$$

The total number of tangents of f is either $\sigma_1\tau$ or $\frac{1}{2}\rho_1(P_{2p-1}-\tau)$. Equating these values we get

$$\tau^2-(2^{2p}-2)\tau+(2^{2p}-1)(2^{2p-2}-1)=0,$$

$$\tau_E=2^{p-1}[2^p+1]-1, \qquad \tau_O=2^{p-1}[2^p-1]-1.$$

Set

(32) $$E_p=2^{p-1}[2^p+1], \qquad O_p=2^{p-1}[2^p-1].$$

For a quadric E with $E_p - 1$ quadric points,

$$\begin{aligned} &\rho_0 = 2^{p-2} P_{p-2} = O_{p-1}, && \sigma_1 = 2^{p-2} P_{p-2} = O_{p-1}, \\ &\rho_1 = P_{2p-3}, && \sigma_2 = 2^{2p-2}, \\ &\rho_2 = E_{p-1}, && \sigma_3 = E_{p-1} - 1. \end{aligned} \tag{33}$$

For a quadric O with $O_p - 1$ quadric points,

$$\begin{aligned} &\rho_0 = E_{p-1}, && \sigma_1 = E_{p-1}, \\ &\rho_1 = P_{2p-3}, && \sigma_2 = 2^{2p-2}, \\ &\rho_2 = O_{p-1}, && \sigma_3 = O_{p-1} - 1. \end{aligned} \tag{34}$$

If $\pi_0, \pi_1, \pi_2, \pi_3$, denote respectively the total number of skew lines, tangents, secants, and generators, of a quadric, then

For a quadric E,

$$\begin{aligned} \pi_0 &= \tfrac{1}{3}\rho_0 [P_{2p-1} - \tau_E] = \tfrac{1}{3} O_p O_{p-1}, \\ \pi_1 &= \tfrac{1}{2}\rho_1 [P_{2p-1} - \tau_E] = \sigma_1 \tau_E = \tfrac{1}{2} P_{2p-3} O_p = O_{p-1}[E_p - 1], \\ \pi_2 &= \rho_2 [P_{2p-1} - \tau_E] = \tfrac{1}{2}\sigma_2 \tau_E = O_p E_{p-1} = 2^{2p-3}(E_p - 1), \\ \pi_3 &= = \tfrac{1}{3}\sigma_3 \tau_E = \tfrac{1}{3}(E_p - 1)(E_{p-1} - 1). \end{aligned} \tag{35}$$

For a quadric O,

$$\begin{aligned} \pi_0 &= \tfrac{1}{3}\rho_0 [P_{2p-1} - \tau_O] = \tfrac{1}{3} E_p E_{p-1}, \\ \pi_1 &= \tfrac{1}{2}\rho_1 [P_{2p-1} - \tau_O] = \sigma_1 \tau_O = \tfrac{1}{2} P_{2p-3} E_p = E_{p-1}(O_p - 1), \\ \pi_2 &= \rho_2 [P_{2p-1} - \tau_O] = \tfrac{1}{2}\sigma_2 \tau_O = E_p O_{p-1} = 2^{2p-3}(O_p - 1), \\ \pi_3 &= = \tfrac{1}{3}\sigma_3 \tau_O = [O_p - 1][O_{p-1} - 1]. \end{aligned} \tag{36}$$

If $\phi = f + C(x^2, y)$, then φ contains the points of both or neither of f and $C(x^2, y)$. If f is an E quadric and y a point not on it, φ contains

$$P_{2p-3} + [O_p - (P_{2p-2} - P_{2p-3})] = O_p - 1$$

points, and φ is an O quadric. If f is an O quadric and y a point not on it, ϕ contains

$$P_{2p-3} + [E_p - (P_{2p-2} - P_{2p-3})] = E_p - 1$$

points, and is an E quadric. Hence

(37) *The 2^{2p} quadrics which belong to C divide into a set of $E_p = 2^{p-1}(2^p+1)$ quadrics E, each containing $E_p - 1$ points; and a set of $O_p = 2^{p-1}(2^p - 1)$ quadrics O each containing $O_p - 1$ points. The number of skew lines, tangents, secants, and generators, of these quadrics, and the number of similar lines on a quadric point or an outside point are furnished by the formulæ* (33), $\cdots$, (36).

It is convenient to use Q, $\bar{Q}$ to denote at the same time E, O and O, E. The following relations among the numbers defined above are sometimes useful:

$$(38)\qquad \begin{aligned} 2Q_{p-1}(\bar{Q}_p - 1) &= P_{2p-3}\,Q_p\ , & Q_p\,\bar{Q}_p &= 2^{2p-2}\,P_{2p-1}, \\ Q_p(Q_p - 1) &= 2P_{2p-1}\,Q_{p-1}, & Q_{p-1}\,\bar{Q}_p &= 2^{2p-3}(Q_p - 1). \end{aligned}$$

A quadric associated with the null system C in (13) is

$$(39)\qquad q_p = x_1 x_{p+1} + x_2 x_{p+2} + \cdots + x_p x_{2p} = 0.$$

Of the p terms in q_p each takes the value 0 in 3 ways, namely, 00, 01, 10; the value 1 in one way, 11. But $q_p \equiv 1$ if $2k+1$ terms take the value 1 and $p - 2k - 1$ terms take the value 0 which occurs in $\binom{p}{2k+1}\,3^{p-2k-1}$ ways. Hence the number of points *not* on q_p is

$$\sum_k \binom{p}{2k+1}\,3^{p-2k-1} = \tfrac{1}{2}\left[(3+1)^p - (3-1)^p\right] = \tfrac{1}{2}\left[2^{2p} - 2^p\right] = O_p.$$

Hence q_p is an E quadric. The quadric

$$q_p + \sum_{i=1}^{2p} a_i x_i^2 \equiv q_p + \left[\sum_1^{2p} a_i x_i\right]^2$$

is of the same type as q_p provided $\Sigma a_i x_i$ is the tangent space of a point on q_p, i. e., if $\sum_{j=1}^{p} a_j a_{j+p} \equiv 0$. Hence

(40) *The 2^{2p} quadrics obtained by varying the a's in*

$$\sum_{j=1}^{p} x_j x_{j+p} + \sum_{i=1}^{2p} a_i x_i^2 = 0,$$

which belong to the null system

$$C = \sum_{j=1}^{p} (x_j x'_{j+p} - x_{j+p} x'_j) = 0,$$

are of the type E or O according as $\sum_{j=1}^{p} a_j a_{j+p} \equiv 0$ or $\equiv 1$.

Other canonical forms of a quadric are useful. Let

$$y_1 y'_1 + y_2 y'_2 + \cdots + y_{2p+1} y'_{2p+1} = 0$$

be the equation of C referred to one of its self dual bases. The quadric

$$\Sigma\, y_i y_k = 0 \qquad (i, k = 1, \cdots, 2p+1;\ i < k),$$

has the polar system C. Any two points of the basis are on a line with a third point all of whose coördinates except two are zero; call these *the residual*

points of the basis. Evidently no residual point is on the quadric. Conversely if a quadric contains no residual point it is the above quadric. For, using $y_2, \cdots, y_{2p+1}$ as a reference basis, the residual points become the reference points and the points with only two coördinates not zero. Since the reference points are not on the quadric every square appears. Also every product term appears, else the corresponding residual point is on the quadric. But

$$(y_2^2 + \cdots + y_{2p+1}^2) = (y_2 + \cdots + y_{2p+1})^2 = y_1 (y_2 + \cdots + y_{2p+1}),$$

and the quadric has the given form.

To find the number of points on the quadric and therefore its kind, let $2k$ be the number of coördinates of a point which are not zero. The point is on the quadric if $2k(2k-1)/2$ is even, i. e., if k is even. Then the total number of points on the quadric is $\sum_{l=1}^{g} \binom{2p+1}{4l}$, where g is the greatest integer for which $4g < 2p+1$. Since

$$4\sum_{l=0}^{g} \binom{2p+1}{4l} = (1+1)^{2p+1} + (1+i)^{2p+1} + (1-1)^{2p+1} + (1-i)^{2p+1},$$

we find that

$$\sum_{l=0}^{g} \binom{2p+1}{4l} = \tfrac{1}{4}\{2^{2p+1} + (1+i)^{2p+1} + (1-i)^{2p+1}\} - 1.$$

If $p = 4n + m$ this reduces to

$$2^{2p-1} + 2^{p-1}\left[\frac{(1+i)(2i)^m + (1-i)(-2i)^m}{2^{m+1}}\right] - 1,$$

i. e., to $E_p - 1$ if $p \equiv 0, 3 \pmod 4$; and to $O_p - 1$ if $p \equiv 1, 2 \pmod 4$. If p is even the basis points are on the quadric.

(**41**) *A basis, $y_1, \cdots, y_{2p+1}$, self dual under C, determines uniquely a quadric*

$$Q = \sum y_i y_k = 0 \quad (i, k = 1, \cdots, 2p+1;\ i < k),$$

belonging to C, which contains none of the residual points of the basis. If $p \equiv 0 \pmod 2$, Q contains the basis points; if $p \equiv 1 \pmod 2$, the lines of the basis are skew to Q. If $p \equiv 0, 3 \pmod 4$, Q is an E quadric; if $p \equiv 1, 2 \pmod 4$, an O quadric.

§4. *The Theta Characteristics as Quadrics in S_{2p-1} Modulo 2 Belonging to C.*

If ω_ϵ is the half period whose Per. Char. is ϵ, and E is a proper exponential factor,* then $E\vartheta(v + \omega_\epsilon)$, considered as a function of v, is a theta function

* K., p. 240. Formulæ (*a*), (*b*), (*c*) are given by K., p. 242; formula (*d*), p. 247.

$\vartheta[\epsilon]_2(v)$, whose *theta characteristic*, or *Th. Char.*, is ϵ. In particular the zero Per. Char. gives rise to the original theta function whose Th. Char. therefore is $\epsilon_i = 0$. Two functions whose Th. Char. are congruent modulo 2 arise from two similar Per. Char. and are not essentially distinct. There are then 2^{2p} Th. Char. including the zero Th. Char. The function with Th. Char. ϵ is *even* or *odd* according as

$$\sum_{\mu=1}^{p} \epsilon_\mu \epsilon_{p+\mu} \equiv 0 \text{ or } \equiv 1 \pmod 2.*$$

Under integral linear transformation of the periods,

$$(a) \qquad \bar{\omega}_{\mu\alpha} = \sum_{\beta=1}^{2p} c_{\alpha\beta}\, \omega_{\mu\beta} \quad (\mu = 1, 2, \cdots, p;\ \alpha = 1, 2, \cdots, 2p),$$

the Per. Char. are transformed as follows:

$$(b) \qquad \epsilon_\beta = \sum_{\alpha=1}^{2p} c_{\alpha\beta}\, \bar{\epsilon}_\alpha \qquad (\beta = 1, 2, \cdots, 2p).$$

The coefficients $c_{\alpha\beta}$ are such that $\sum_{\mu=1}^{p} (\epsilon_\mu \eta_{p+\mu} - \epsilon_{p+\mu} \eta_\beta)$ is invariant, i. e.,

$$(c) \qquad \sum_{\mu=1}^{p} (c_{\mu\beta}\, c_{p+\mu,\gamma} - c_{\mu\gamma}\, c_{p+\mu,\beta}) \equiv \begin{matrix} 1 \text{ if } \gamma = p+\beta, \\ 0 \text{ if } \gamma \neq p+\beta. \end{matrix}$$

The 2^{2p} functions $\vartheta[\epsilon]_2(v)$ are transformed, to within exponential factors, into a similar system $\bar{\vartheta}[\bar{\epsilon}]_2(\bar{v})$, the Th. Char. of the two systems being connected by the equations

$$(d) \qquad \begin{aligned} \bar{\epsilon}_\nu &= \sum_{\mu=1}^{p} (c_{\nu\mu}\, \epsilon_\mu - c_{\nu,p+\mu}\, \epsilon_{p+\mu} + c_{\nu\mu}\, c_{\nu,p+\mu}), \\ \bar{\epsilon}_{p+\nu} &= \sum_{\mu=1}^{p} (-c_{p+\nu,\mu}\, \epsilon_\mu + c_{p+\nu,p+\mu}\, \epsilon_{p+\mu} + c_{p+\nu,\mu}\, c_{p+\nu,p+\mu}) \end{aligned} \qquad (\nu = 1, 2, \cdots, p).$$

Since these equations hold as congruences modulo 2 and since $c_{\alpha\beta}^2 \equiv c_{\alpha\beta}$ and $-1 \equiv 1 \pmod 2$, we can modify them so as to read

$$(42) \qquad \bar{\epsilon}_\beta = \sum_{\mu=1}^{p} (c_{\beta,\mu}\, c_{\beta,p+\mu} + c_{\beta\mu}^2\, \epsilon_\mu + c_{\beta,p+\mu}^2\, \epsilon_{p+\mu}) \quad (\beta = 1, 2, \cdots, 2p).$$

In § 2 we have identified the points, $x_1, x_2, \cdots, x_{2p}$, of S_{2p-1} (mod 2) with the Per. Char. ϵ; and the collineations $x_\beta = \sum_{\alpha=1}^{2p} c_{\alpha\beta}\, \bar{x}_\alpha$ of S_{2p-1} which leave the null system, $C = \sum_{\mu=1}^{p} (x_\mu x'_{p+\mu} - x_{p+\mu} x'_\mu)$, unaltered with the transformations

* K., p. 240. Formulæ (*a*), (*b*), (*c*) are given by K., p. 242; formula (*d*), p. 247.

(*b*) of the Per. Char. Consider the effect of such a collineation upon the quadric

$$\sum_{\mu=1}^{p} (x_\mu x_{p+\mu} + \epsilon_\mu x_\mu^2 + \epsilon_{p+\mu} x_{p+\mu}^2), \tag{43}$$

which belongs to C. It must be transformed into another of the same sort, say with coefficients $\bar{\epsilon}$. By effecting the collineation upon the quadric and making use of the relations (*c*) the coefficients $\bar{\epsilon}$ of the transformed quadric turn out to be those defined by (42). Hence by making use of (40), we obtain the second fundamental theorem:

(**44**) *Under integral linear transformation of the periods the 2^{2p} Th. Char. are permuted just as the 2^{2p} quadrics in S_{2p-1} modulo 2 which belong to C are permuted under the collineation group G_{NC} of C. The theta function with given Th. Char. is odd or even according as the corresponding quadric is an O or an E quadric. Thus the parity of the characteristic is invariant under such transformation.**

According to the formula

$$\vartheta [\epsilon]_2 (v + \omega_\eta) = E \vartheta [\epsilon + \eta]_2 (v),$$

the function $\vartheta [\epsilon]_2 (v)$ vanishes when v is ω_η if $\vartheta [\epsilon + \eta]_2 (v)$ is an odd function. Regarding $\vartheta [\epsilon]_2 (v)$ as a quadric Q and ω_η, as a point P, then $\vartheta [\epsilon + \eta]_2 (v)$ is the quadric Q' obtained by adding to Q the square of the null S_{2p-2} of P. If Q' is an O quadric then, either Q is an E quadric and P is not on Q or Q is an O quadric and P is on Q. We have therefore a further translation scheme:

(45)

An E (or O) quadric which belongs to C.	A theta function with an even (or odd) Th. Char.
An E quadric does (or does not) contain a given point.	An even theta function does not (or does) vanish for a half period with given Per. Char.
An O quadric does (or does not) contain a given point.	An odd theta function does (or does not) vanish for a half period with given Per. Char.

The number‡ of odd and of even thetas is gotten from (37), while the enumerations contained in (32), $\cdots$, (36) characterize very fully the behavior of a particular theta with regard to the sets of three syzygetic or three azygetic Per. Char.

* Cf. the proof of the invariance of parity given by K., pp. 247–50.

† K., p. 240 (VII).

‡ K., p. 252.

§ 5. *The Period and Theta Characteristics as a Linear System in R_{2p} with Reference to a Quadric. Projection and Section Applied to Quadrics. Steiner and Kummer Sets.*

We have already remarked that the sum of two quadrics belonging to C is the square of an S_{2p-2}. More generally, a sum of a number of quadrics and a number of squared S_{2p-2}'s is a quadric or a squared S_{2p-2} according as the number of quadrics in the sum is odd or even.* Thus the 2^{2p} quadrics and $2^{2p} - 1$ squared S_{2p-2}'s lie in a linear system containing $2^{2p+1} - 1$ elements, i. e., in a linear space R_{2p}. A concrete representation of this linear system is obtained by mapping the points of S_{2p-1} on the points of a quadric M in R_{2p} by means of $2p + 1$ independent quadrics in S_{2p-1} belonging to C. Taking the convenient canonical form of (40), let

$$z_0 = \sum_{j=1}^{p} x_j \cdot x_{p+j}, \qquad z_i = x_i^2 \qquad (i = 1, 2, \cdots, 2p). \tag{46}$$

Then the points x of S_{2p-1} are mapped on the points z of

$$M = z_0^2 + \sum_{j=1}^{p} z_j z_{p+j} = 0. \tag{47}$$

M is the general type of non-degenerate quadric in R_{2p}. The collineation group which leaves it unaltered is simply isomorphic with G_{NC}.† The null system of M is necessarily degenerate whence there is one point, $z_0 = 1$, $z_i = 0$, whose polar R_{2p-1} as to M is evanescent. This point we shall call the *vertex* V of M. Since C connects a point of S_{2p-1} with its null S_{2p-2} or also with the square of its null S_{2p-2} we can identify the period characteristics with the squared S_{2p-2}'s and thus show that

(**48**) *The Per. Char. and Th. Char. can be represented as the linear system of $2^{2p+1} - 1$ R_{2p-1}'s in a space R_{2p} with reference to a proper quadric M. The R_{2p-1}'s on the vertex V of M correspond to the Per. Char.; those not on V correspond to the Th. Char., which are odd or even according as the R_{2p-1} cuts M in an O or an E quadric.*

An R_{2p-1} on V cuts M in a quadric section with a double point z on M. The point z is the map of a point x in S_{2p-1} whose null S_{2p-2} corresponds to the R_{2p-1}. This is the trace in R_{2p} of the null system C.

The above representation with reference to M in R_{2p} will be retained only in the background for purposes of suggestion, the important feature being the linearity of the entire system of characteristics. As an instance of the usefulness of this feature let x be a given point in S_{2p-1}. It is a linear condition that a quadric or S_{2p-2} be on x, whence there are $2^{2p} - 1$ quadrics and S_{2p-2}'s

*K, p. 254.

† DICKSON, *Linear Groups*, p. 197, §§ 199, 200.

and $2^{2p-1}-1$ S_{2p-2}'s on x. There are then 2^{2p-1} quadrics on x. These quadrics are paired by the involution, I_x, the members of a pair having the same section by the null S_{2p-2} of x. Such a set of $2\cdot 2^{2(p-1)}$ quadrics will be called* *a first Steiner set* or *the Steiner set of* x. Again if $x^{(1)}$ and $x^{(2)}$ are two points of a null line a quadric or S_{2p-2} on both must contain the third point of the line. This imposes two linear conditions on the quadrics and S_{2p-2}'s whence there are $2^{2p-1}-1$ quadrics and S_{2p-2}'s and $2^{2p-2}-1$ S_{2p-2}'s on the line. Hence there are $4\cdot 2^{2(p-2)}$ quadrics on the line which divide into $2^{2(p-2)}$ sets of 4, each set of 4 having the same section by the null S_{2p-3} of the null line $\overline{x^{(1)}x^{(2)}}$. The members of a set of 4 are permuted transitively by the Abelian G_4 generated by the involutions $I_{x^{(1)}}$ and $I_{x^{(2)}}$. Such a set of $2^2\cdot 2^{2(p-2)}$ quadrics will be called a *second Steiner set* or *the Steiner set of the null line* $\overline{x^{(1)}x^{(2)}}$. Evidently the argument can be carried on to the limit set by the null S_{p-1} and we can say generally that

(**49**) *A null* S_{m-1} $(m=1,\cdots,p)$ *determines an* m*th Steiner set of* $2^m\cdot 2^{2(p-m)}$ *quadrics which contain* S_{m-1}. *They divide into* $2^{2(p-m)}$ *sets of* 2^m *quadrics. Each set of* 2^m *quadrics has the same section by the null* S_{2p-m-1} *of* S_{m-1} *and its members are permuted regularly by the Abelian* G_{2^m} *generated by the involutions* I_x *of the points* x *of a reference basis of* S_{m-1}.†

Further properties of the quadrics of an mth Steiner set can be derived from the projection and section of C from S_{m-1} and within S_{2p-m-1} as explained after (23). We have then a derived space $\Sigma_{2\pi-1}$, $\pi=p-m$, whose points correspond to S_m's on S_{m-1} and within S_{2p-m-1}. Evidently a set of 2^m quadrics on S_{m-1} which have the same section by S_{2p-m-1} determine in $\Sigma_{2\pi-1}$ a single quadric. Hence the above $2^{2\pi}$ sets of 2^m quadrics determine in $\Sigma_{2\pi-1}$ the $2^{2\pi}$ quadric belonging to the null system Γ_π. If a point of $\Sigma_{2\pi-1}$ lies on a quadric, the corresponding S_m on S_{m-1} lies on each of the corresponding set of 2^m quadrics in S_{2p-1}. For $m=1$, we see from the values of σ_3 in (33) and (34) that the section of a quadric of type Q is a quadric of the same type. Since the general projection and section can be effected by projections and sections from successive points it is clear that the type of quadric is unaltered in the process. Thus from (37) for $p=\pi$ we find that

(**50**) *The* $2^{2(p-m)}$ *sets of* 2^m *quadrics in the* m*th Steiner set of a null* S_{m-1} *divide into* O_{p-m} *sets of* 2^m O *quadrics and* E_{p-m} *sets of* 2^m E *quadrics.*

For the important particular case, $m=1$, the enumerations contained in (32) to (36) lead to the following results in S_{2p-1}:

(**51**) *Given a* Q *quadric and a line* l *tangent to* Q *at* x; *there are* $\bar{Q}_{p-2}$ *planes on* l *containing two further tangents at* x, P_{2p-5} *planes on* l *containing a further tangent and generator on* x, *and* Q_{p-2} *null planes on* l *containing two further*

* When $p=3$, the odd quadrics of the set correspond to the well known Steiner complex of 12 double tangents of a plane quartic curve.

† For $m=1,2,3$, cf. K., pp. 255–65.

generators on x; given a generator m of Q on x, there are $\bar{Q}_{p-2}$ planes on m containing two further tangents at x, 2^{2p-4} planes on m containing a further tangent and generator on x, and $Q_{p-2}-1$ null planes on m containing two further generators on x. The number of planes on x containing respectively three tangents, two tangents and a generator, a tangent and two generators, and three generators, of Q on x is $\frac{1}{3}\bar{Q}_{p-2}\bar{Q}_{p-1}$, $\bar{Q}_{p-2}[Q_{p-1}-1]$, $Q_{p-2}\bar{Q}_{p-1}$, and $\frac{1}{3}(Q_{p-1}-1)(Q_{p-2}-1)$.

Call a null S_{m-1} contained in Q *a generator* S_{m-1} *of* Q; and a null S_m which meets Q in a null S_{m-1} *a tangent* S_m *of* Q. Then after projection and section from S_{m-1} on Q, the earlier enumerations lead to the following results:

(52) *Given a generator S_{m-1} of Q, and a tangent S_m containing it; there are on S_m $\bar{Q}_{p-m-1}$ S_{m+1}'s containing two other S_m's tangent along S_{m-1}, $P_{2(p-m)-3}$ S_{m+1}'s containing another S_m tangent along S_{m-1} and a generator S_m, and Q_{p-m-1} null S_{m+1}'s containing two generator S_m's; given a generator S_m on S_{m-1}, there are on S_m, $\bar{Q}_{p-m-1}$ S_{m+1}'s containing two S_m's tangent along S_{m-1}, $2^{2(p-m-1)}S_{m+1}$'s containing a generator S_m and a tangent S_m, and $Q_{p-m-1}-1$ null S_{m+1}'s containing two other generator S_m's. On S_{m-1}, the number of S_{m+1}'s containing respectively three tangent S_m's, two tangents S_m's and a generator S_m, a tangent S_m and two generator S_m's, and there generator S_m's is $\frac{1}{3}\bar{Q}_{p-m}\bar{Q}^{*}_{p-m-1}$, $\bar{Q}_{p-m-1}(Q_{p-m}-1)$, $\bar{Q}_{p-m}Q_{p-m-1}$, and $\frac{1}{3}[Q_{p-m}-1][Q_{p-m-1}-1]$.*

Some fairly obvious deductions from the above general theorems can now be drawn; e. g., from (50), for $m=1$, it is clear that,

(53) *An S_{2p-2} can be expressed in O_{p-1} ways as a sum of two O quadrics; in E_{p-1} ways as a sum of two E quadrics; and in 2^{2p-2} ways as a sum of an O and an E quadric. Or, if a squared S_{2p-2} be added to the 2^{2p} quadrics, then of the Q_p Q quadrics, $2Q_{p-1}$ become Q quadrics while the remaining 2^{2p-2} become $\bar{Q}$ quadrics.**

Two points determine a third on their join. If the join is a null line every quadric contains only one or all of the three points; those containing all constituting the second Steiner set of the line. If the join is an ordinary line a quadric on one point must contain a second point but cannot contain all three. In particular a pair on one of the points do not meet in another of the three. Calling the first Steiner sets of the points syzygetic or azygetic according as the points are syzygetic or azygetic [see (14)], we have

(54) *A first Steiner set is determined uniquely by any one of its Q_{p-1} pairs of Q quadrics. Two first Steiner sets determine another first Steiner set, the three being symmetrical. If the three are syzygetic, they together contain all the quadrics and have in common a second Steiner set; i. e., the sets have $4Q_{p-2}$ Q quadrics in common, while $3\cdot 2^{2p-3}$ Q quadrics occur in only one set. If the three are azygetic, they have no common quadrics, any two have Q_{p-1} Q quadrics in com-*

* K., p. 258 (VI) and (VII).

mon, but have no pairs in common. Together the three contain $3Q_{p-1}$ Q *quadrics, and* $\bar{Q}_{p-1}$ Q *quadrics are found in none.**

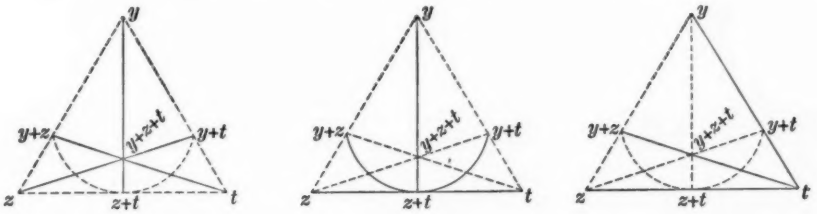
Applying (54) to the projection from S_{m-1} and translating the result from $\Sigma_{2\pi-1}$ back to S_{2p-1}, we obtain the theorem:

(55) *If three null* S_m*'s on a null* S_{m-1} *are in an ordinary* S_{m+1}*, no quadric contains all three but any two are contained in* $2^m Q_{p-m-1}$ Q *quadrics; if the three are in a null* S_{m+1} *every quadric on* S_{m-1} *contains at least one, while* $3 \cdot 2^m 2^{2(p-m)-3}$ Q *quadrics contain only one and* $4 \cdot 2^m \cdot Q_{p-m-2}$ Q *quadrics contain all three.*

In (54) we have considered the Steiner sets of three points of a line. Let us suppose the three points form a triangle. To contain the points is three independent linear conditions on a quadric or S_{2p-2} whence there are $2^{2p-2} - 1$ quadrics and S_{2p-2}'s and $2^{2p-3} - 1$ S_{2p-2}'s on the three points or

(56) *The Steiner sets of three points which form a triangle have* 2^{2p-3} *quadrics in common.*

Four cases are possible according as 0, 1, 2, or 3, of the three sides of the triangle are null lines. Drawing null lines full, we see from the figure of Case I



that a quadric on y, z, t cannot contain $y + z + t$, else the three null lines are generators and therefore also the sides contrary to the hypothesis that they are ordinary lines. The null space S of $y + z + t$ contains the triangle and is a secant space of all quadrics on the triangle. If S be added to all these quadrics each Q quadric on the triangle becomes a $\bar{Q}$ quadric on the triangle whence the number of Q and $\bar{Q}$ quadrics is the same. Thus from (56) we find

(57) *The Steiner sets of three points which form a triangle with ordinary sides have in common* 2^{2p-4} Q *quadrics unpaired in each set. Each* Q *quadric meets a definite* $\bar{Q}$ *quadric on the null* S_{2p-2} *of the null subspace of the plane of the triangle.*†

In case IV the plane of the triangle is a null plane and every quadric on the triangle contains the plane whence

(58) *The first Steiner sets of three points which form a triangle with null sides have in common the third Steiner set of the plane of the triangle.*‡

* K., p. 260 (VIII), p. 261 (IX).

† Cf. K., p. 263 (X).

‡ Cf. K., p. 265 (XIII).

From the figure in case II we see that a quadric on y, z, t contains two null lines on $z + t$, but not the third, and conversely. Hence theorem (55), for $m = 1$, can be applied. In case III a quadric on y, z, t contains two null lines on t but not the third, and the same theorem applies. Hence

(59) *The Steiner sets of three points which form a triangle with one or with two null sides have in common* $2\ Q_{p-2}$ *Q quadrics. In the first case the quadrics are not paired in any set though they are paired in the Steiner set of the third point of the null side. In the second case the quadrics are paired in the Steiner set of the point on the two null sides.**

Some of these theorems are easily generalized. Thus (58) and the first case of (54) are special cases of the following:

(60) *If any number of null spaces lie in a null* S_{m-1} *their Steiner sets have in common the mth Steiner set of the null* S_{m-1}.

Following out (57), let $x^{(1)}, \cdots, x^{(2k+1)}$ be an odd number of points of a self dual basis of C. They lie in an S_{2k} which has an S_0, $x^{(1)} + \cdots + x^{(2k+1)}$, as a null subspace. As in (56), there are $2^{2p-2k-1}$ quadrics on the points which are interchanged in type by adding the null S_{2p-2} of S_0 which is secant to all. Hence

(61) *The Steiner sets of an odd number,* $2k + 1$, *of points of a self dual basis of C have in common* $2^{2p-2k-2}$ *Q quadrics which are unpaired in each set.*

If however we have an even number 2ρ of points of a self dual basis of C as in the second case of (54), their $S_{2\rho-1}$ has no null subspace and the above argument does not apply. $S_{2\rho-1}$ determines its null space $S_{2(p-\rho)-1}$, which also has no null subspace; in fact, each is the null space of the other and the two are skew. Let Γ_ρ be the section of C by $S_{2\rho-1}$ [as defined after (22)], $\Gamma_{p-\rho}$ the section of C by $S_{2(p-\rho)-1}$. These two sections define C. For if x is a point of $S_{2\rho-1}$, its null $S_{2\rho-2}$ under Γ_ρ together with $S_{2(p-\rho)-1}$ are contained in an S_{2p-2}, the null space of x under C; similarly, if y is a point of $S_{2(p-\rho)-1}$. If z is a point of neither, the $S_{2\rho}$ joining z to $S_{2\rho-1}$ meets $S_{2(p-\rho)-1}$ in a point y, the line yz meets $S_{2\rho-1}$ in a point x and the null S_{2p-2} of z is determined from those of x and y; the line yzx is of course a null line. If a reference basis of S_{2p-1} be chosen by taking a reference basis x of $S_{2\rho-1}$ and a reference basis y of $S_{2(p-\rho)-1}$, a quadric belonging to C can have only product terms in x and product terms in y. Since the squared terms also are separable the quadric is determined by its two sections. Hence

(62) *If* $S_{2\rho-1}$ *and* $S_{2(p-\rho)-1}$ *are skew null spaces of each other under C, and if* Γ_ρ *and* $\Gamma_{p-\rho}$ *are their sections of C, then C is determined by the two sections. If Q′ is a quadric in* $S_{2\rho-1}$ *belonging to* Γ_ρ, *Q″ a quadric in* $S_{2(p-\rho)-1}$ *belonging to* $\Gamma_{p-\rho}$ *there is a single quadric Q belonging to C which contains the sections Q′ and Q″. Q is an E quadric if Q′ and Q″ are of the same type, an O quadric if Q′ and Q″ are of different types.*

* K., p. 264 (XI), p. 265 (XII).

The last statement is proved as follows: Any line xy is a null line and therefore a tangent or generator of Q. If x is on Q', y on Q'' it is a generator and contains a point z of Q. If x is on Q' and y not on Q'' it must be a tangent to Q at x. If x is not on Q' and y is on Q'' it must be a tangent to Q'' at y. If x is not on Q' and y not on Q'' it must be a tangent to Q at z. Since every point z of Q is on one such line, the number of points on Q if Q' and Q'' are of the same sort is

$$(Q_\rho - 1)(Q_{p-\rho} - 1) + (Q_{p-\rho} - 1) + (Q_\rho - 1) + \bar{Q}_\rho \bar{Q}_{p-\rho} = E_p - 1$$

and Q is an E quadric. If Q' and Q'' are unlike the number of points on Q is

$$(Q_\rho - 1)(\bar{Q}_{p-\rho} - 1) + (Q_\rho - 1) + (\bar{Q}_{p-\rho} - 1) + \bar{Q}_\rho Q_{p-\rho} = O_p - 1$$

and Q is an O quadric.

The above suggests for $\rho = 1$ an obvious construction for the null system C and its quadrics in S_{2p-1} when a null system and its quadrics in S_{2p-3} are given.

According to (62), the 2^{2p} quadrics Q are determined by pairing the $2^{2\rho}$ quadrics Q' with the $2^{2(p-\rho)}$ quadrics Q''. If Q has the section Q' every quadric $Q + S^2$ where S is a null S_{2p-2} on S_{2p-1} has the same section. S then must be the null S_{2p-2} of a point y on $S_{2(p-\rho)-1}$. If y lies on Q'', $Q + S^2$ is of the same type as Q, otherwise of a different type. Thus

(63) *If $S_{2\rho-1}$ has no null subspace and if $E(\rho)$ is an E quadric belonging to the section of C by $S_{2\rho-1}$, there are $E_{p-\rho}$ E quadrics and $O_{p-\rho}$ O quadrics containing E_ρ; if $O(\rho)$ is an O quadric belonging to the section of C by $S_{2\rho-1}$ there are $O_{p-\rho}$ E quadrics and $E_{p-\rho}$ O quadrics containing $O(\rho)$.*

For $\rho = 1$ we have the second case in (54), since then a quadric $E(\rho)$ is a pair of points and a quadric $O(\rho)$ has no real points.

We have therefore a method for determining the number and kind of quadrics on a group of points which lie in a space which has no null subspace. Taking up again the case of 2ρ points, azygetic in pairs, which lie in $S_{2\rho-1}$ we first find how many quadrics $Q(\rho)$ are on these points. Taking the points as a reference basis in $S_{2\rho-1}$ the quadric on them must be

$$\Sigma x_i x_k = 0 \qquad (i, k = 1, \cdots, 2\rho;\ i < k).$$

To find the type of this quadric we note that a point lies on it if $4r$ or $4r + 1$ of its coördinates are not zero. The number of its points is $\binom{2\rho}{1} + \binom{2\rho}{4} + \binom{2\rho}{5} + \cdots$. Adding and subtracting $\binom{2\rho}{0} = 1$, this number is obtained from the expansions of $(1 + i^s)^{2\rho}$, $s = 0, 1, 2, 3$, and is

$$\tfrac{1}{4}\{2(1+1)^{2\rho} + (1-i)(1+i)^{2\rho} + (1+i)(1-i)^{2\rho}\} - 1;$$

i. e., $E_\rho - 1$ if $\rho \equiv 0, 1 \pmod 4$ and $O_\rho - 1$ if $\rho \equiv 2, 3 \pmod 4$. Hence from (63),

(64) *The Steiner sets of 2ρ points of a self dual basis of C have in common $Q_{p-\rho}$ Q quadrics if $\rho \equiv 0, 1 \pmod 4$; and $\bar{Q}_{p-\rho}$ Q quadrics if $\rho \equiv 2, 3 \pmod 4$.*

When $\rho = 1$ we obtain again the second case of (54). Theorems (61) and (64) exhaust the cases arising from points of a self dual basis of C.

The above canonical form of a quadric is of especial interest for $\rho = p$. It is determined by $2p$ points of a basis and must coincide with the quadric of (41) when p is even.

(65) *If the $2p$ points of the reference basis x of S_{2p-1} belong to a self dual basis of C, the unique quadric belonging to C and on the $2p$ basis points*

$$\Sigma x_i x_k = 0$$

coincides with the quadric (41) *when p is even. When p is odd the quadric does not contain the $(2p+1)$th basis point nor the residual points of the reference basis, but does contain the remaining residual points of the basis. It is an E or an O quadric according as $p \equiv 0, 1$ or $p \equiv 2, 3 \pmod 4$.*

We define a *Kummer set of quadrics* to be all the quadrics which do *not* pass through a point. The Kummer and Steiner sets of a point exhaust the 2^{2p} quadrics. A *Caporali set of quadrics* consists of all the quadrics which are skew to an ordinary line.* By analogy with the Steiner sets, a Caporali set might be called a second Kummer set. But the analogy could be carried no further, since any S_k, $k > 1$, meets every quadric in real points.

Quadrics and squared S_{2p-2}'s constitute a linear system, R_{2p}; squared S_{2p-2}'s constitute a linear system S_{2p-1}. The quadrics common to the Kummer sets of r linearly independent points satisfy r inequalities and by the use of (3) we see that

(66) *The Kummer sets of r linearly independent points have 2^{2p-r} quadrics in common.*

An obvious argument from (54) shows that

(67) *The Kummer set of a point contains 2^{2p-2} Q quadrics. The Kummer sets of two syzygetic points have 2^{2p-3} Q quadrics in common, of three points on a null line no quadrics in common. The Kummer sets of two or three points on an ordinary line have $\bar{Q}_{p-1}$ Q quadrics in common which constitute a Caporali set.*

Let us consider the Kummer sets of three points which form a triangle taking up the Cases I, $\cdots$, IV defined above. In case I, a quadric common to the three sets must contain $y + z + t$ and that point only. By projection from the point we ask for the quadrics not on the three points of an ordinary line, i. e., a Caporali set of the projected space. We originally had therefore $2 \cdot \bar{Q}_{p-2}$ Q quadrics. In case IV, the common quadrics contain the null line $y + z$,

* Cf., for the nomenclature: TIMERDING, *Ueber die Gruppirungen der Doppeltangenten einer ebenen Curve vierter Ordnung*, Journal für Mathematik, vol. 122 (1900), p. 209, where the terms are analogously defined for odd quadrics, $p = 3$.

$z+t, t+y$, and are projected from this line into the Kummer set of a point whence originally there were

$$4 \cdot 2^{2(p-2)-2} = 2^{2p-4}$$

Q quadrics. In case II, the common quadrics contain $z+t$ and are projected from this point into a Caporali set so that originally there were $2 \cdot \bar{Q}_{p-2}$ Q quadrics. In case III, a quadric not on y, z, t must be on $y+t, z+t, y+z+t$ and we thus get again the case of (57) with the same result as our present case IV. Hence

(**68**) *The Kummer sets of the vertices of a triangle have in common* 2^{2p-3} *quadrics. These consist of* $2\ \bar{Q}_{p-2}$ Q *quadrics if none or one of the sides of the triangle are null lines, of* 2^{2p-4} Q *quadrics if two or three sides of the triangle are null lines.*

Case IV is easily generalized. Let $x^{(1)}, \cdots, x^{(m)}$ be a reference basis of a null S_{m-1}. A quadric common to their Kummer sets must contain a null S_{m-2} in S_{m-1} without containing S_{m-1}. Projected from S_{m-2}, S_{m-1} becomes a point and the quadric belongs to the Kummer set of the point; whence

(**69**) *The Kummer sets of the points of a reference basis of a null* S_{m-1} *have in common* 2^{2p-m-1} Q *quadrics which divide into sets of* 2^{m-1}.

To determine the quadrics common to the Kummer sets of a number of basis points we can utilize a new canonical form of the quadric. Take for reference point basis $2p$ points of a basis of C and let $x_1, \cdots, x_{2p}$ be the coördinates. Any quadric not on the $2p$ points must contain every squared term and, the pairs of points being azygetic, must also contain every product term and therefore is $q = \sum_{i=1}^{2p} x_i^2 + q'$, where

$$q' = \Sigma x_i x_k \qquad (i, k = 1, \cdots, 2p;\ i < k).$$

Since Σx_i is the polar S_{2p-2} of the unit point as to q' and this point lies on q' if p is even, q is of the same type as q' if p is even. Or, from (65), q is an E or an O quadric according as $p \equiv 0, 3$ or $\equiv 1, 2 \pmod 4$.

(**70**) *If the* $2p$ *points of the reference basis* x *of* S_{2p-1} *belong to a self dual basis of* C, *the unique quadric not on the* $2p$ *basis points*

$$\sum_{i=1}^{2p} \Sigma x_i^2 + \Sigma x_i x_k = 0 \qquad (i, k = 1, \cdots, 2p;\ i < k)$$

is the quadric (41) *when* p *is odd. When* p *is even, the quadric does not contain the residual points of the reference basis but does contain the* $(2p+1)$*th basis point and the remaining residual points. It is an* E *or an* O *quadric according as* $p \equiv 0, 3$ *or* $\equiv 1, 2$ *(mod* 4).

Given then an even number 2ρ of basis points. They lie in an $S_{2\rho-1}$ without

a null subspace. A quadric not on these points is cut by $S_{2\rho-1}$ in a quadric $Q(\rho)$ not on these points. According to (70) there is a unique quadric $Q(\rho)$ of this type. From $Q(\rho)$ we pass to the original quadric as in (63) and find that

(71) *The Kummer sets of* 2ρ *points of a self dual basis of* C *have in common* $Q_{p-\rho}$ Q *quadrics if* $\rho \equiv 0, 3 \pmod 4$, *and* $\bar{Q}_{p-\rho}$ Q *quadrics if* $\rho \equiv 1, 2 \pmod 4$.

$2\rho + 1$ points of a self dual basis lie in an $S_{2\rho}$ which has a null subspace S_0 and any quadric not on the $2\rho + 1$ points is on S_0. By projection and section from S_0 we obtain the case of (71) and have therefore shown that

(72) *The Kummer sets of* $2\rho + 1$ *points of a self dual basis of* C *have in common* 2 $Q_{p-\rho-1}$ Q *quadrics if* $\rho \equiv 0, 3 \pmod 4$, *and* 2 $\bar{Q}_{p-\rho-1}$ Q *quadrics if* $\rho \equiv 1, 2 \pmod 4$.

The Steiner and Kummer sets have received considerable attention in the particular case, $p = 3$. Numerous other sets are suggested by the smaller values of p and can be readily generalized and discussed by the foregoing methods. In the next paragraph a somewhat different point of view is emphasized and again the geometrical method of treatment seems most effective.

§ 6. *Systems of Quadrics.*

A linear system F_r of quadrics is determined by $r + 1$ linearly independent quadrics $Q, Q_1, \cdots, Q_r$. The sum of any even number of the quadrics is the square of an S_{2p-2} which has a null point y. Let $Q + Q_i$ have the null point $y^{(i)}$. Then F_r determines an S_{r-1} with the reference basis $y^{(1)}, \cdots, y^{(r)}$. Moreover any reference basis of S_{r-1} together with any quadric of F_r determines F_r. Let us call S_{r-1} the *allied space of* F_r. Two systems, F_r and F'_r, with the same allied space and a common quadric coincide. Thus the 2^{2p} quadrics can be divided in a single way into 2^{2p-r} systems F_r with a given common allied space S_{r-1}. These 2^{2p-r} systems F_r are called a *complex allied with* S_{r-1}. Two systems, F_r and F_s, are *skew systems* or *null systems of each other* if their allied spaces are respectively skew spaces or null spaces of each other. Two skew systems may or may not have one common quadric. The two cases can be distinguished by the respective terms *partially skew* or *completely skew*. A *Göpel system* F_r has for allied space a null S_{r-1}; a *Göpel system* is allied with a Göpel space S_{p-1}.*

There are two types of system F_0, namely, an E quadric and an O quadric. There are three types of system F_1, a pair of E quadrics, a pair of O quadrics, and an E and O quadric. The allied space S_0 is on the quadrics in the first two types but not in the last.

Let a system F_2 be determined by the quadric Q and the allied space S_1

* Cf., for these definitions, K., p. 296, § 9.

with points $y, z, y+z$. The system contains four quadrics,

$$Q, \quad Q + C(x^2, y), \quad Q + C(x^2, z), \quad Q + C(x^2, y+z),$$

whose sum is identically zero. Any three of the four determine the system. According as the line S_1 is skew to, tangent to, secant to, or on, Q, three, two, one, or none, of the quadrics are $\bar{Q}$ quadrics. That is, the set of four contains an even number of each kind if S_1 is a null line, an odd number of each kind if S_1 is an ordinary line. In the first case we say that any three of the four are syzygetic;* in the second case any three of the four are azygetic.* There are then five types of system F_2: three syzygetic types with $4, 2, 0, E$ quadrics respectively; and two azygetic types with $3, 1, E$ quadrics respectively. In the first and third types S_1 is a generator of the four quadrics; in the second type S_1 touches one pair at one point, the other pair at another point; in the fourth and fifth types S_1 is skew to the one quadric and cuts the other three in two out of three of its points.

Given two quadrics of the same type, Q, $Q' = Q + C(x^2, y)$, where y is on Q, the number of pairs which can be added to the given pair to form a syzygetic or azygetic system is determined by the numbers $\sigma_1, \sigma_2, \sigma_3$, [(33) and (34)]. If the given pair are of opposite type, $Q, \bar{Q}$, the numbers ρ_0, ρ_1, ρ_2, serve. Hence

(**73**) *Given two quadrics Q, Q', there are $\bar{Q}_{p-1}$ pairs $\bar{Q}$, $\bar{Q}'$, and $Q_{p-1} - 1$ pairs Q'', Q''', each syzygetic with the two and 2^{2p-2} pairs Q'', $\bar{Q}$, each agyzetic with the two; given two quadrics $Q, \bar{Q}$, there are P_{2p-3} pairs, $Q', \bar{Q}'$, each syzygetic with the two, and $\bar{Q}_{p-1}$ pairs $\bar{Q}', \bar{Q}''$, and Q_{p-1} pairs Q', Q'', each azygetic with the two.*

Similarly, the numbers $\pi_0, \pi_1, \pi_2, \pi_3$ [(35) and (36)] serve to determine the number of sets of three quadrics syzygetic or azygetic with one given quadric.

(**74**) *A Q quadric is syzygetic with $\bar{Q}_{p-1}(Q_p - 1)$ triads $Q', \bar{Q}, \bar{Q}'$ and with $\frac{1}{3}(Q_p - 1)(Q_{p-1} - 1)$ triads Q', Q'', Q''', azygetic with $\frac{1}{3}\bar{Q}_p \bar{Q}_{p-1}$ triads $\bar{Q}, \bar{Q}', \bar{Q}''$, and with $2^{2p-3}(Q_p - 1)$ triads $Q', Q'', \bar{Q}$.*

An obvious enumeration and the use of (38) leads to the following result:

(**75**) *There are $\frac{1}{12} Q_p (Q_p - 1)(Q_{p-1} - 1) = \frac{1}{3} P_{2p-1} P_{2p-3} Q_{p-2}$ syzygetic tetrads Q, Q', Q'', Q'''; $\frac{1}{2} Q_p \bar{Q}_{p-1} (Q_p - 1) = 2^{2p-4} P_{2p-1} P_{2p-3}$ syzygetic tetrads $Q, Q', \bar{Q}, \bar{Q}'$; and $\frac{1}{3} 2^{2p-2} P_{2p-1} Q_{p-1}$ azygetic tetrads $Q, Q', Q'', \bar{Q}$.*

Let $s^{(1)}, \cdots, s^{(2p+1)}$, be a self dual basis of C and let $Q^{(0)}$ be any quadric. $Q^{(0)}$ and the $2p+1$ quadrics $Q^{(i)} = Q^{(0)} + S^{(i)2}$ are subject to the single relation $\sum_{i=0}^{2p+1} Q^{(i)} = 0$. Any three of the quadrics are azygetic if $S^{(i)} + S^{(k)}$, $S^{(k)} + S^{(l)}$, $S^{(l)} + S^{(i)}$ do not form a null pencil. $S^{(i)} + S^{(k)}$ and $S^{(k)} + S^{(l)}$

* K., p. 253.

are in an ordinary pencil if

$$(S^{(i)}+S^{(k)},\ S^{(k)}+S^{(l)})=(S^{(i)},\ S^{(k)})+(S^{(i)},\ S^{(l)})+(S^{(k)},\ S^{(l)})\neq 0.$$

But by the definition of the basis, $(S^{(i)}, S^{(k)}) \neq 0$. Such a set of $2p+2$ quadrics any three of which are azygetic is called a *fundamental set* and is denoted by F.S. If the set be given and one be added to the others the squares of $2p+1$ S_{2p-2}'s are obtained which must form a basis because of the azygetic property of the quadrics. Thus a basis determines 2^{2p} F.S.'s while an F.S. is determined from $2p+2$ bases. The number of sets then is $N_{BC}\,2^{2p}/(2p+2)$.

(**76**) *A fundamental set, F.S., of $2p+2$ quadrics (a set such that all are connected by one linear relation and any three are azygetic) is obtained by adding one quadric to the squared S_{2p-2}'s of a basis of C. If any quadric of a F.S. be added to the others a self dual basis of C is obtained. The number N_F of F.S.'s is**

$$N_F=\frac{2^{2p+p^2}(2^{2p}-1)(2^{2p-2}-1)\cdots(2^2-1)}{(2p+2)!}=\frac{2^{p(p+2)}}{(2p+2)!}P_{2p-1}P_{2p-3}\cdots P_1.$$

In (41) we showed that with a self dual basis $y_1, \cdots, y_{2p+1}$, there is associated a unique quadric

$$R_y=\Sigma\, y_i y_k \qquad (i,\ k=1,\ \cdots,\ 2p+1,\ i<k).$$

From the basis and R_y we can construct a definite F.S. which has certain special properties and which will be called a *normal* F.S. Since R_y contains the basis points if p is even, the quadrics of the normal F.S. are then all of the same type. When p is odd R_y is the only one of its type.

(**77**) *The normal F.S. determined by the basis y of C and its quadric R_y contains only E quadrics if $p=0$ (mod 4); contains only O quadrics if $p\equiv 2$ (mod 4); contains the O quadric R_y and $2p+1$ E quadrics if $p\equiv 1$ (mod 4); and contains the E quadric R_y and $2p+1$ O quadrics if $p\equiv 3$ (mod 4). Moreover a F.S. which contains $2p+1$ quadrics of the same type is a normal F.S. for which the remaining quadric is R_y.*†

The last statement is proven as follows: Let Q, $Q+y_i^2$, $i=1, 2, \cdots, 2p+1$, be a F.S. such that the quadrics $Q+y_i^2$ are all of the same type. The null points of y_1 and y_2 are both on or both off Q according to the type of $Q+y_i^2$. Their line is an ordinary line since any three of the quadrics are azygetic and is either a secant or skew line of Q. In either case the residual point, the null point of y_1+y_2 is not on Q, and Q is the quadric R_y associated with the basis.

Any F.S., Q, $Q+y_i^2$, determines not only the basis y but also the $2p+1$ bases gotten by varying k in y_k, y_k+y_i $(i\neq k)$. This set of $2p+2$ bases

* Cf. K., pp. 283–5.

† Cf. K., pp. 274–6; also p. 288.

is a symmetrical set consisting of any one basis and its residual points. The set is transformed into itself by a $G_{(2p+2)!}$, the symmetric group on the $2p+2$ bases, which is generated by the $(p+1)(2p+1)$ involutions I_x on the point x belonging to the bases. Sample involutions are $y_i' = y_i$, $y_k' = y_k + y_i$ $(i \neq k)$ and $y_i' = y_k$, $y_k' = y_i$, $y_l' = y_l$ $(l \neq i,\ l \neq k)$. The quadric R_y attached to one basis is also attached to the others if p is odd since according to (41) it contains neither the basis points nor the residual points. But if p is even, the basis points are on R_y and the residual points are not, whence there are $2p+2$ quadrics R_y.

(**78**) *The points and residual points of a basis form a basis configuration, i. e., a set of* $(p+1)(2p+1)$ *points which can be divided into a basis and its residual points in* $2p+2$ *ways, each point lying in two bases. The configuration is unaltered by a* $G_{(2p+2)!}$ *generated by the involutions on its points, which is symmetric on its bases. Any F.S. determines a basis configuration.*

If p is even, the basis configuration determined by a normal F.S. contains bases whose quadrics R_y make up the normal F.S. But if p is odd the basis configuration of the normal F. S. determines the unique quadric R_y, isolated in the normal F.S. R_y and the basis configuration determine $2p+2$ normal F.S.'s containing $(p+1)(2p+1)$ quadrics (each quadric in two F.S.'s) apart from R_y which occurs in each F.S.

(**79**) *If* p *is even a normal F.S. contains the* $2p+2$ *quadrics attached to the bases of the configuration determined by the F.S. and is invariant under the configuration* $G_{(2p+2)!}$. *If* p *is odd, a normal F.S. is one of a set of* $2p+2$ *normal F.S.'s each having the same isolated quadric and basis configuration. The configuration* $G_{(2p+2)!}$ *permutes the* $2p+2$ *F.S.'s symmetrically, each F.S. being invariant under the* $G_{(2p+1)!}$ *attached to its particular basis.*

For example, when $p = 2$ the 15 points of S_3 form a single basis configuration containing the 6 bases in S_3. There is but one normal F.S., consisting of the 6 O quadrics attached to the 6 bases. When $p = 3$, a normal F.S. containing one E quadric and 7 O quadrics determines a basis configuration. This configuration contains 8 bases and arises from 8 normal F.S.'s each having the same E quadric. That is, the Aronhold sets of seven can be grouped in 36 ways, corresponding to the even characteristics, into 8 sets, each odd characteristic occurring in two sets.

The following enumeration is immediate:

(**80**) *If* $p \equiv 0$ *or* 3 *(mod* 4*), for each* E *quadric there are* N_{BC}/E_p *self dual bases of* C *whose lines are secant or skew lines respectively of* E. *If* $p \equiv 2$ *or* 1 *(mod* 4*) there are* N_{BC}/O_p *self dual bases of* C *whose lines are secant or skew lines respectively of* O. *If* p *is odd the bases can be grouped into basis configurations.*

The normal F.S. affords a convenient method for studying the general F.S.

Let Q, Q_1, $\cdots$, Q_{2p+1}, where $Q+Q_i=y_i$, be any F.S. Since $\sum_1^{2p+1} y_i^2=0$, any S_{2p-2} can be expressed as $\sum_{i=1}^{2k} y_i^2$, where $k=1, \cdots, p$. Hence Q itself can be written as, $Q=R_y+\sum^{2k} y_i^2$, and the F.S. takes the form

$$Q=(R_y+\sum^{2k} y_i^2),\quad Q_1=(R_y+\sum^{2k} y_i^2)+y_1^2,\quad \cdots,\quad Q_h=(R_y+\sum^{2k} y_i^2)+y_h^2,\quad \cdots.$$

Q and R_y are of the same or different type according as k is even or odd. The quadrics Q_h divide into two sets, according as y_h^2 does or does not occur in $\sum^{2k} y_i^2$. Hence we have

(α) 1 quadric Q of the form $R_y+\sum^{2k} y_i^2$,

(β) $2k$ quadrics of the form $R_y+\sum^{2k-1} y_i^2=R_y+\sum^{2p-2k+2} y_j^2$,

(γ) $2p+1-2k$ quadrics of the form $R_y+\sum^{2k+1} y_i^2=R_y+\sum^{2p-2k} y_j^2$.

Quadrics (α), (β), (γ) are of the type R if respectively k is even, $p-k$ is odd, $p-k$ is even. Thus we have the table:

Residue of p modulo 4	R of type	Number of O quadrics	
		for k even	for k odd
0	E	$2k$	$2(p-k)+2$
1	O	$2k+1$	$2(p-k)+1$
2	O	$2(p-k)+2$	$2k$
3	E	$2(p-k)+1$	$2k+1$

Hence if s is the number of O quadrics, $s \equiv p \pmod 4$. Moreover, for any such number s, a F.S. containing s O quadrics can be constructed by determining k in the table from the given s. The required F.S. can be written down in terms of the basis and $2k$ arbitrarily selected S_{2p-2}'s of it. Since any two bases are conjugate and within a basis any two sets of $2k$ spaces are conjugate, then any two F.S.'s with the same number s are conjugate, provided that the same F.S. can be determined from k even or k odd—a double possibility for the same s. Denote by y' the basis

$$y_1'=y_1,\qquad y_i'=y_1+y_i\qquad (i=2;\ \cdots,\ 2p+1).$$

If p is odd let y_1 be not contained in $\sum^{2k} y_i^2$. Then $R_y=R_{y'}$, $\sum^{2k} y_i^2=\sum^{2k} y_i'^2$, and the above F.S. is

$$Q_1=(R_{y'}+\sum^{2k+1} y_i'^2),\quad Q=(R_{y'}+\sum^{2k+1} y_i'^2)+y_1'^2,\quad Q_2=(R_{y'}+\sum^{2k+1} y_i'^2)+y_2'^2,\quad \cdots.$$

But

$$\sum^{2k+1} y_i'^2 = \sum^{2(p-k)} y_j'^2.$$

Hence, if p is odd, k can be replaced by $p-k$. If p is even, let y_1 be in $\sum^{2k} y_i^2$. Then

$$R_y = R_{y'} + y_1'^2, \qquad \sum^{2k} y_i^2 = \sum^{2k-1} y_i'^2,$$

and the original F.S. is

$$Q_1=(R_{y'}+\sum^{2k-1} y_i'^2),\quad Q=(R_{y'}+\sum^{2k-1} y_i'^2)+y_1'^2,\quad Q_2=(R_{y'}+\sum^{2k-1} y_i'^2)+y_2'^2,\quad \cdots.$$

Since $\sum^{2k-1} y_i'^2 = \sum^{2(p-k)+2} y_j'^2$, if p is even, k can be replaced by $p-k+1$. Thus the same F.S. can be gotten from an even or an odd k. If Q and Q_i are of the same type, the involution, I_x, determined by $Q+Q_i$, interchanges Q and Q_i and leaves all the other quadrics unaltered. No collineation other than the identity can leave every quadric unaltered.

(**81**) *A F.S. contains s O quadrics, where $s \equiv p \pmod 4$. If $s \equiv p \pmod 4$, there are $N/\{s!\,(2p+2-s)!\}$ F.S.'s containing s O quadrics, all conjugate under G_N and each invariant under a subgroup $G_{s!\,(2p+2-s)!}$ of G_N. This subgroup is the product of the interchangeable groups $G_{s!}$ and $G_{(2p+2-s)!}$ the symmetric groups on the O and E quadrics respectively of the F.S. The subgroup has an invariant quadric or an invariant space (the sum of the O or E quadrics in F.S.), according as p is odd or even. It has also two invariant skew spaces S_{s-2} and S_{2p-s}, the allied spaces of the O and of the E quadrics. The F.S. can be obtained from a normal F.S. by adding a squared S_{2p-2}.**

Let S_{p-1} be a Göpel space, F_p a Göpel system. If F_p contains two quadrics, Q and $\bar{Q}$, of different types, any third quadric Q' of F_p is syzygetic with the two and therefore paired with a fourth $\bar{Q}'$ of different type, so that $Q+\bar{Q}+Q'+\bar{Q}'=0$. Hence F_p contains 2^{p-1} E quadrics and 2^{p-1} O quadrics. If however the quadrics of F_p are all of the same type, they all must contain S_{p-1}. Since there are, on a null S_k, 2^{k+1} Q_{p-k-1} Q quadrics [60], and $E_0=1$ while $O_0=0$, S_{p-1} is on 2^p E quadrics and those only. Hence

(**82**) *In every complex of 2^p Göpel systems there is one Göpel system which contains only E quadrics; each of the 2^p-1 other Göpel systems contains 2^{p-1} Q quadrics. Each set of 2^{p-1} Q quadrics has an allied null space S_{p-2} which lies on every quadric of the set. Any three quadrics in a system are syzygetic.*†

A precisely similar argument applies to the case where the S_{m-1} allied to the system F_m is a null S_{m-1}. If F_m contains a quadric of each type half of the 2^m quadrics in F_m are Q quadrics. If all the quadrics are of one type they

* Cf. K., pp. 286–9; XXVIII–XXXI.

† K., p. 300, XXXIV.

all contain S_{m-1}. There are 2^m Q_{p-m} Q quadrics containing S_{m-1} and these divide in a unique way into sets of 2^m with the allied space S_{m-1}.

(**83**) *If S_{m-1} is a null space, the complex of 2^{2p-m} F_m's with the allied space S_{m-1} contains Q_{p-m} systems F_m containing only Q quadrics. The members of these systems contain S_{m-1}. The remaining $2^{2(p-m)}$ P_{m-1} systems F_m contain 2^{m-1} quadrics of each type.**

Denote by $\phi_{Q,m}$ the system F_m containing quadrics of the type Q only. Such a system becomes, after projection and section from S_{m-1}, a single quadric of type Q in the derived space $\Sigma_{2\pi-1}$, $\pi = p - m$. From the definition of syzygetic and azygetic quadrics in $\Sigma_{2\pi-1}$ we have

(**84**) *Three systems $\varphi_{Q,m}$ allied with the null S_{m-1} are contained in a system F_{m+2} allied with an S_{m+1} on S_{m-1}. The three systems are syzygetic or azygetic according as S_{m+1} is a null or an ordinary space. The entire theory here developed of the quadrics Q belonging to C in S_{2p-1}, when applied to the quadrics belonging to C_π in $\Sigma_{2\pi-1}$ yields an analogous theory of the systems $\varphi_{Q,m}$ in S_{2p-1}.*

This theorem is due essentially to FROBENIUS† though its remarkable utility, exemplified in many of the preceding theorems, is clearly apparent only when it is viewed as a result of the general process—projection and section from a null space.

Let us now investigate the "section C_ρ of C by the space $S_{2\rho-1}$ which has no null subspace"; i. e., by *the Rosenhain space* $S_{2\rho-1}$. Let us call a system of quadrics, $F_{2\rho}$, with the allied Rosenhain space, $S_{2\rho-1}$, *a Rosenhain system.*‡ A Rosenhain space, $S_{2\rho-1}$, determines its *complementary* skew Rosenhain space, $S_{2(p-\rho)-1}$, the two being null spaces of each other. According to (62) and (63),

(**85**) *The $2^{2\rho}$ quadrics of a Rosenhain system $F_{2\rho}$ allied with an $S_{2\rho-1}$ have the same section $Q(p-\rho)$ by the complementary Rosenhain space $S_{2(p-\rho)-1}$. They divide into E_ρ Q quadrics and O_ρ $\bar{Q}$ quadrics. The complex of $2^{2(p-\rho)}$ Rosenhain systems with the allied $S_{2\rho-1}$ contains $E_{p-\rho}$ systems with E_ρ E quadrics and O_ρ O quadrics and $O_{p-\rho}$ systems with E_ρ O quadrics and O_ρ E quadrics.*

Denoting by $\psi_{E,\rho}$ and $\psi_{O,\rho}$ these respective systems we obtain the theorem analogous to (84):

(**86**) *Three Rosenhain systems $\psi_{Q,\rho}$ allied with the Rosenhain space $S_{2\rho-1}$ are contained in a system $F_{2\rho+2}$ allied with an $S_{2\rho+1}$ which cuts the complementary Rosenhain space $S_{2(p-\rho)-1}$ in an S_1. The three systems are syzygetic or azygetic according as $S_{2\rho+1}$ is not or is a Rosenhain space; or also according as S_1 is not or is an ordinary line, i. e., a Rosenhain S_1. The theory developed above of the quadrics Q belonging to C in S_{2p-1}, applied to the quadrics belonging to $C_{p-\rho}$ in $S_{2(p-\rho)-1}$, yields an analogous theory of the systems $\psi_{Q,\rho}$ in S_{2p-1}.*

* K., p. 303, XXXVI.

† Cf. K., pp. 302–5, where references are given.

‡ For $p = 2$, $\rho = 1$, cf. K., pp. 337–8.

Let us determine finally the number of E and O quadrics in the general system F_r with allied space S_{r-1}. Let S_{r-1} have the null subspace S_{m-1} with reference basis $x^{(1)}, \cdots, x^{(m)}$. Let S_{r-m-1} be a space skew to S_{m-1} in S_{r-1} with reference basis $y^{(m+1)}, \cdots, y^{(r)}$ which is part of a self dual basis of C. According to (20), $r - m$ is even and S_{r-m-1} is a Rosenhain space. F_r is fixed by means of S_{r-1} and any one of its quadrics Q. If Q contains S_{m-1} so also does every quadric of F_r. Then either all or none of the quadrics of F_r contain S_{m-1}. In the latter case Q and S_{m-1} determine a system F_m and by adding to F_m the null spaces of the $2^{r-m} - 1$ points in S_{r-m-1}, 2^{r-m} systems F_m are obtained each consisting [see (83)] of 2^{m-1} Q quadrics. The systems F_m exhaust the system F_r, whence F_r contains 2^{r-1} Q quadrics. In case, however, Q contains S_{m-1}, the system F_r becomes by projection and section from S_{m-1}, a system F_{r-m} with an allied Rosenhain space derived from the projection of S_{r-m-1}. In S_{2p-1} the complementary Rosenhain space of S_{r-m-1} is an $S_{2p-(r-m)-1}$ which has in common with the null S_{2p-m-1} of S_{m-1} an $S_{2p-m-(r-m)-1}$ which contains S_{m-1} and a skew space $S_{2(p-m)-(r-m)-1}$. This last space and S_{r-m-1} itself project from S_{m-1} into complementary Rosenhain spaces in $\Sigma_{2\pi-1}$. Then theorem (85) can be applied to the system F_{r-m} in $\Sigma_{2\pi-1}$ and the result translated to S_{2p-1}. Hence

(**87**) *A space S_{r-1} with null subspace S_{m-1} is determined by the Göpel space S_{m-1} and a skew Rosenhain space S_{r-m-1}. The complementary Rosenhain space cuts the null space of the Göpel space in an $S_{2p-m-(r-m)-1}$ which can be determined by the Göpel space and a skew $S_{2(p-m)-(r-m)-1}$. By projection and section from the Göpel space, S_{r-m-1} and $S_{2(p-m)-(r-m)-1}$ become complementary Rosenhain spaces.*

*A system F_r determined by Q' and S_{r-1} contains 2^{r-1} Q quadrics if Q' does not contain S_{m-1}. If Q' contains S_{m-1} and meets $S_{2(p-m)-(r-m)-1}$ in a quadric of type Q'', then F_r contains $2^m E_{(r-m)/2}$ quadrics of type Q'' and $2^m O_{(r-m)/2}$ quadrics of type $\bar{Q}''$.**

KRAZER remarks (p. 266) that the Per. and Th. Char. have been confused by various writers. He notes (pp. 253–4) some points of difference between the two but does not call express attention to the fact—fundamental in the exposition here given—that the coefficients of the transformation occur *linearly* in the transformation of the Per. Char. and *quadratically* in the transformation of the Th. Char. Though we find (p. 254) that "die Summe einer geraden Anzahl von Th. Char. sich wie eine Per. Char., die Summe einer ungeraden Anzahl von Th. Char. aber wie eine Th. Char. transformiert," yet it is stated (p. 284) that "Die Summe der $2p + 2$ Th. Char. eines F.S. is [0]," i. e., is the zero Th. Char.; and (p. 305) that "Durch . . . der Addition einer beliebigen Th. Char. zu den sämtlichen Th. Char. eines Systems geht ein System von Th. Char. immer wieder in ein System von Th. Char. über."

* Cf. K., p. 301, XXXV.

Consider also the statement (p. 270): "Man fasse nun die (Per.) charakteristiken des F.S. als Th. Char. . . . auf." A Per. Char. can be regarded as a Th. Char. only through the intervention of some given Th. Char. which Krazer implicitly takes to be the zero Th. Char. However if entire accuracy is sought in the use of such a process one should state whether the resulting theorems are or are not independent of the given Th. Char. employed for the transition, i. e., whether the results are covariant under G_N or covariant only under the subgroup of G_N defined by the given Th. Char. Such distinctions or limitations are almost self evident from the geometrical point of view.

BALTIMORE,
June 1, 1912.

CONFORMAL TRANSFORMATIONS ON THE BOUNDARIES OF THEIR REGIONS OF DEFINITION*

BY

WILLIAM F. OSGOOD AND EDSON H. TAYLOR

Riemann's problem of mapping a simply connected plane region whose boundary consists of more than a single point conformally on a circle as normal region may be divided into two parts: (*a*) the internal problem; namely, the map of the interior points,† and (*b*) the boundary problem; namely, the behavior of the map on the boundary.

The first of these problems was treated by RIEMANN in his dissertation with the aid of Dirichlet's principle, and rigorous proofs were supplied for regions with restricted boundaries by SCHWARZ and NEUMANN, the general case being established by OSGOOD through methods due to POINCARÉ.

The second problem was solved for analytic boundaries by Schwarz, the case of vertices being disposed of by PICARD.‡ PAINLEVÉ§ treated the case of arcs of boundaries that are convex regular curves, and his results can be extended by mere linear transformations to any boundaries consisting of arcs of regular curves with continuous curvature. For wholly arbitrary boundaries the problem of the continuity of the map on the boundary was considered by OSGOOD‖ in a series of theorems. It is the purpose of the present paper to give proofs of these theorems and to supplement them.

The propositions of § 5,—more particularly, Lemmas 2 and 3, for in Lemma 1 it is only the uniqueness of the function that is new,—were suggested by intuition, a harmonic function being thought of as the temperature in a flow of heat or the potential in a flow of electricity.

In the comprehensiveness of the results here obtained, applying as they do to the most general boundaries that consist of a single piece and more than one point, lies, it is believed, an intrinsic interest. So many problems of

* Presented to the Society December 30, 1902 and April 30, 1910.

† For an account of the first problem and the second in the case of analytic boundaries, with references to the literature, cf. OSGOOD, *Lehrbuch der Funktionentheorie*, vol. 1, 2nd ed., 1912, chap. 14.

‡ PICARD, *Traité d'analyse*, vol. 2, chap. 10, § 7.

§ Comptes Rendus, vol. 112 (1891), p. 653.

‖ Bulletin of the American Mathematical Society, vol. 9 (1903), p. 233.

mathematics must be restricted in order to be treated at all, that it is a source of satisfaction when the most general result of its class can be established.

§ 1. *Line and Surface Integrals.*

Let the functions u, u_1, u_2 be harmonic in a finite region S of the (x, y)-plane, whose boundary is denoted by C. Under suitable further restrictions pertaining to the boundary and the behavior of the functions there, the following equations are familiar.*

$$(A) \qquad \int_S \int \left(\frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial y} \right) dS = - \int_C u_2 \frac{\partial u_1}{\partial n} ds;$$

$$(B) \qquad \int_S \int \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right] dS = - \int_C u \frac{\partial u}{\partial n} ds;$$

$$(C) \qquad \int_C u_1 \frac{\partial u_2}{\partial n} ds = \int_C u_2 \frac{\partial u_1}{\partial n} ds;$$

$$(D) \qquad \int_C \frac{\partial u}{\partial n} ds = 0.$$

We shall need these equations for a somewhat more general case than the one ordinarily treated. It is sufficient for our purposes to assume that C consists of a finite number of arcs, each analytic inclusive of its extremities. Let P be a vertex of C, and denote the angle at P by α; then shall $0 < \alpha < 2\pi$.

The functions u, u_1, u_2 shall satisfy the following conditions:

1) u, u_1, u_2 are finite and harmonic within S;

2) u, u_1, u_2 are continuous on the boundary;

3) the first partial derivatives of u, u_1, u_2 are continuous on the boundary except at one or more of the vertices. In the neighborhood of such a vertex, P, each of these derivatives remains less numerically than $Cr^{-\mu}$, where r denotes the distance of (x, y) from P, where C and μ are constants, and where $0 < \mu < 1$:

$$\left| \frac{\partial u_1}{\partial x} \right| < \frac{C}{r^\mu}, \text{ etc.}$$

Under these conditions it is readily shown that each of the above integrals which is an improper integral converges. Let S' be the part of S that remains after a small neighborhood of each vertex P has been cut off by an arc γ of a circle of radius h and center P. All four equations hold for the region S'.

* In all line integrals involving ds, which appear in this paper, ds is taken as a positive quantity. Cf. the definition laid down by Osgood, *Funktionentheorie*, vol. 1, chap. 4.

Moreover, it is seen that the contributions of the above line integrals due to the arcs γ all approach 0 simultaneously with h, and this establishes the four equations under the given conditions.

§ 2. *A Lemma.*

LEMMA. *Let u satisfy the following conditions:*

1) *u is finite and harmonic within S;*

2) $$u\,|_L = 0,$$

where L denotes an arc of the boundary of S abutting on a vertex P, the relation holding at all points of L except P;

3) $$\left.\frac{\partial u}{\partial n}\right|_{L'} = 0,$$

*the relation holding at all points of L' except P.**

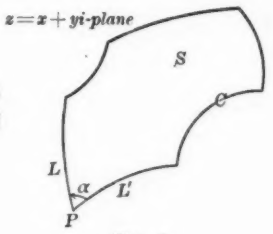


FIG. 1.

Then

$$\left|\frac{\partial u}{\partial x}\right| < \frac{C}{r^{\mu}}, \qquad \left|\frac{\partial u}{\partial y}\right| < \frac{C}{r^{\mu}},$$

where r denotes the distance of (x, y) from P, where C is a positive constant, and where $\mu = 1 - \frac{1}{2}\nu$, $\nu = \pi/\alpha$. Moreover, u is harmonic along L'.

Let the interior of S be transformed conformally on the upper half of the $w = \xi + \eta i = \rho e^{\theta i}$-plane by means of the function

$$w = f(z), \tag{1}$$

the point P going over into the point O: $w = 0$. Then u, considered as a function of ξ, η, is finite and harmonic throughout the interior of a semicircle K_1: $|w| < R_0$, $0 < \eta$; and if R_0 be suitably restricted, u vanishes along the radius OQ_1: $\theta = \pi$, $0 < \rho \leqq R_0$, while $\dfrac{\partial u}{\partial n} = \dfrac{1}{\rho}\dfrac{\partial u}{\partial \theta}$ vanishes along the radius OQ_2: $\theta = 0$, $0 < \rho \leqq R_0$. Hence u is capable of harmonic continuation by reflection across OQ_1 into the lower half-plane, and u, thus extended, is seen to be harmonic throughout the domain K: $0 < \rho < R_0$, $0 < \theta < 2\pi$.

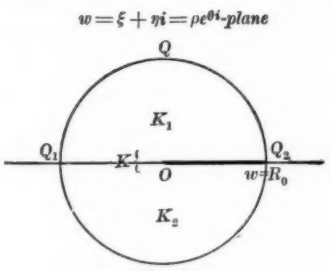


FIG. 2.

Consider the function

$$v = \frac{\partial u}{\partial \theta}.$$

* It is assumed throughout the paper, when the condition is imposed that a normal derivative vanish, that the first partial derivatives of the function are continuous along the piece of the boundary in question except at the extremities of the arc.

This function is harmonic in K, as follows by substitution in Laplace's equation. Moreover, v vanishes on each bank of the cut $OQ_2 : \theta = 0$, since along the upper bank

$$0 = \frac{\partial u}{\partial n} = \frac{1}{\rho}\left(\frac{\partial u}{\partial \theta}\right)_{\theta=0} \qquad (0 < \rho),$$

and along the lower bank

$$\left(\frac{\partial u}{\partial \theta}\right)_{\theta=2\pi-} = \left(\frac{\partial u}{\partial \theta}\right)_{\theta=0+}.$$

Subject the w-plane to the transformation

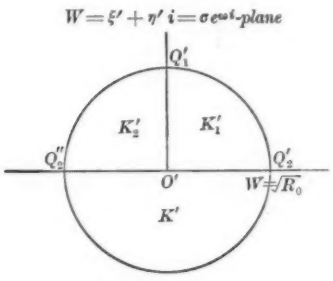


FIG. 3.

$$W = w^{\frac{1}{2}},$$

where

$$W = \sigma e^{\omega i}, \qquad \sigma = \sqrt{\rho}, \qquad \omega = \tfrac{1}{2}\theta.$$

Then v goes over into a function of σ, ω:

$$v = f(\sigma, \omega),$$

harmonic within the semicircle $Q_2'' Q_1' Q_2'$ and vanishing along the diameter $Q_2'' Q_2'$, except perhaps at O'. It is, therefore, possible to continue v by reflection harmonically across this diameter, and we are thus led to a function harmonic throughout the complete circle K': $0 \leqq \sigma < \sqrt{R_0}$ except at O'.

The function v can be developed into a series of the form*

$$v = k \log \sigma + \sum_{n=-\infty}^{\infty} \sigma^n (a_n \cos n\omega + b_n \sin n\omega),$$

where

$$a_n = \frac{1}{\pi a^n}\int_0^{2\pi} V \cos n\omega \, d\omega, \qquad n \neq 0; \qquad b_n = \frac{1}{\pi a^n}\int_0^{2\pi} V \sin n\omega \, d\omega,$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} V d\omega - k \log a, \qquad V = f(a, \omega) \qquad (0 < a < \sqrt{R_0}).$$

We proceed next to show that $a_n = 0$ and that $b_{2m} = 0$. In points of K' situated symmetrically with respect to the axis of ξ', v has values that are equal and opposite, while $\cos n\omega$ has equal values. Hence

$$\int_0^{\pi} V \cos n\omega \, d\omega = -\int_{\pi}^{2\pi} V \cos n\omega \, d\omega$$

and

$$a_n = 0, \qquad n \neq 0; \qquad a_0 = -k \log a.$$

* OSGOOD, *Lehrbuch der Funktionentheorie*, vol. 1, 2nd ed., 1912, p. 660.

Furthermore, in two points of K_1' and K_2' which are situated symmetrically with respect to the axis of η', v has equal values, since such points are the images of points in the w-plane situated symmetrically with respect to the axis of reals, and in these latter points u has values which are equal and opposite. Hence

$$f\left(\sigma, \frac{\pi}{2} - \omega\right) = f\left(\sigma, \frac{\pi}{2} + \omega\right) \qquad (0 < \sigma < \sqrt{R_0}).$$

But

$$\sin n\left(\frac{\pi}{2} - \omega\right) = -\sin n\left(\frac{\pi}{2} + \omega\right)$$

if n is even. Hence, for such values of n,

$$\int_{-\pi/2}^{\pi/2} V \sin n\omega \, d\omega = -\int_{\pi/2}^{3\pi/2} V \sin n\omega \, d\omega,$$

or

$$b_{2m} = 0.$$

The result thus far obtained is, then, this:

$$v = k \log \frac{\sigma}{a} + \sum_{n=-\infty}^{\infty} b_{2n+1} \sigma^{2n+1} \sin (2n+1)\omega.$$

Since $v = 0$ when $\theta = 0$, and hence when $\omega = 0$, it follows that $k = 0$, $a_0 = 0$.

Consider next the function u carried over into the W-plane. Since

$$\frac{\partial u}{\partial \omega} = \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial \omega} = 2\frac{\partial u}{\partial \theta} = 2v,$$

it follows that

$$\frac{\partial u}{\partial \omega} = 2\sum_{n=-\infty}^{\infty} b_{2n+1} \sigma^{2n+1} \sin (2n+1)\omega.$$

The function u vanishes along the line $\omega = \pi/2$. Hence u and its harmonic continuation will be given at any point (σ, ω) of the circle K' distinct from the center by the integral

$$\int_{\pi/2}^{\omega} \frac{\partial u}{\partial \omega} d\omega,$$

σ being held fast. Along this path the above series converges uniformly. Hence

$$u = -2\sum_{n=-\infty}^{\infty} \frac{b_{2n+1}}{2n+1} \sigma^{2n+1} \cos (2n+1)\omega.$$

So much, independent of the condition that u be finite. Imposing this condition, we see that*

* Osgood, *Funktionentheorie*, vol. 1, pp. 647 and 656.

$$b_{2n+1} = 0, \qquad 2n+1 < 0.$$

For, u can have at most a removable singularity at O', and if u be defined there by its limiting value, it will be harmonic there and so admit a unique development of the above form, no negative powers of σ entering.

Finally, returning to the w-plane, we have:

$$u = -2\sum_{n=0}^{\infty} \frac{b_{2n+1}}{2n+1} \rho^{\frac{2n+1}{2}} \cos\frac{2n+1}{2}\theta. \tag{2}$$

From this development it is seen that in the neighborhood of the point O

$$\left|\frac{\partial u}{\partial \xi}\right| < \frac{M}{\rho^{\frac{1}{2}}}, \qquad \left|\frac{\partial u}{\partial \eta}\right| < \frac{M}{\rho^{\frac{1}{2}}}, \tag{3}$$

where M denotes a positive constant. Furthermore, u is harmonic along L'.

To complete the proof we need the theorem of the next paragraph.

§ 3. *An Asymptotic Approximation for the Map near a Vertex of the Boundary.*

THEOREM. *If the function $f(z)$ of § 2 be written in the form*

$$f(z) = z^{\nu}\lambda(z), \qquad \nu = \frac{\pi}{\alpha}, \tag{4}$$

the point $z = 0$ being taken at P, then $\lambda(z)$ approaches a limit not 0 when z, remaining in S, approaches 0 as its limit.

A proof of this theorem may be given by means of the results obtained by KELLOGG.* Let the region S be transformed on a region Σ of the $Z = X + Yi$-plane by the equation

$$Z = z^{\nu}.$$

Through the above function $Z = z^{\nu}$ and the function (1): $w = f(z)$ the region Σ is transformed on the upper half of the $w = \xi + \eta i$-plane, $Z = 0$ going over into $w = 0$. Consider a finite portion T of the latter half-plane bounded by a curve having continuous curvature and including so much of the neighborhood of $w = 0$ as lies in that half-plane. Let T also be so restricted that the boundary Γ of its image Σ' in Σ will have continuous curvature except at $Z = 0$. It is readily shown that Γ satisfies Kellogg's Condition (A), p. 41, of the first paper cited.

Furthermore η, regarded as a function of X, Y, is harmonic within Σ' and is 0 in all the boundary points near $Z = 0$, while its boundary values in all points satisfy Kellogg's Condition (B), l. c., p. 42. It follows, then, from

*O. D. KELLOGG, these Transactions, vol. 9 (1908), p. 39 and p. 51; ibid., vol. 13 (1912), p. 109.

Kellogg's results that the first derivatives of η in Σ' approach continuous boundary values.

Finally, $\partial\eta/\partial X$ and $\partial\eta/\partial Y$ never vanish simultaneously on the boundary. The proof of this fact can be given by the transformation which Kellogg uses in his third paper, p. 122.

Hence it appears that $\Delta w/\Delta Z$, formed for the point $Z=0$, approaches a limit as ΔZ approaches 0, and this limit is not 0. But

$$w = Z\lambda(Z^{1/\nu}), \qquad \text{and} \qquad \frac{\Delta w}{\Delta Z} = \lambda(\Delta Z^{1/\nu}).$$

Thus the theorem is established.

In particular, setting $w=\rho e^{\theta i}$ and $z=re^{\phi i}$, we infer that

$$\rho = r^{\nu}|\lambda(z)|, \tag{5}$$

where

$$0 < A < |\lambda(z)| < B,$$

A and B being constants. Furthermore, for points z within S,

$$\frac{dw}{dz} = \frac{dw}{dZ}\frac{dZ}{dz} = \frac{dw}{dZ}\cdot \nu z^{\nu-1}.$$

Since dw/dZ is finite near $Z=0$ and remains in absolute value greater than a positive constant, and since the same is true of $\nu e^{(\nu-1)\phi i}$, it follows that

$$\frac{dw}{dz} = \frac{\Omega(z)}{r^{1-\nu}},$$

where

$$0 < A' < |\Omega(z)| < B'.$$

Hence

$$\left|\frac{\partial\xi}{\partial x}\right| = \left|\frac{\partial\eta}{\partial y}\right| < \frac{B'}{r^{1-\nu}}, \qquad \left|\frac{\partial\xi}{\partial y}\right| = \left|\frac{\partial\eta}{\partial x}\right| < \frac{B'}{r^{1-\nu}}, \tag{6}$$

and

$$\left|\frac{\partial\xi}{\partial x}\right| + \left|\frac{\partial\xi}{\partial y}\right| = \left|\frac{\partial\eta}{\partial x}\right| + \left|\frac{\partial\eta}{\partial y}\right| > \frac{A'}{r^{1-\nu}}. \tag{7}$$

A simpler and more direct proof of the foregoing theorem is desirable. Kellogg's solution introduces more than is needed, since his determination of the density of the double distribution contains *all* the properties of the function η, while we need only such as are sufficient for the study of the partial derivatives of the first order near the boundary.

§ 4. *Completion of the Proof of the Lemma.*

We desire an upper limit (Abschätzung) for the numerical values of $\partial u/\partial x$, $\partial u/\partial y$ near P. We have:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x},$$

$$\left|\frac{\partial u}{\partial x}\right| \leqq \left|\frac{\partial u}{\partial \xi}\right| \cdot \left|\frac{\partial \xi}{\partial x}\right| + \left|\frac{\partial u}{\partial \eta}\right| \cdot \left|\frac{\partial \eta}{\partial x}\right|,$$

with a similar relation for $\left|\frac{\partial u}{\partial y}\right|$. From (3) and (5) it follows that

$$\left|\frac{\partial u}{\partial \xi}\right|, \qquad \left|\frac{\partial u}{\partial \eta}\right| < \frac{M}{\sqrt{A}} \cdot \frac{1}{r^{\nu/2}}.$$

Furthermore, from (6):

$$\left|\frac{\partial \xi}{\partial x}\right|, \qquad \left|\frac{\partial \eta}{\partial x}\right| < \frac{B'}{r^{1-\nu}}.$$

Hence, finally:

$$\left|\frac{\partial u}{\partial x}\right|, \qquad \left|\frac{\partial u}{\partial y}\right| < \frac{2MB'}{\sqrt{A}} \cdot \frac{1}{r^{1-\nu/2}}, \tag{8}$$

Q. E. D.

Obviously Condition (2) of the lemma can be replaced by the condition: $u\,|_L = k$, where k is any constant, since the function $u - k$ would then satisfy all the conditions of the lemma.

§ 5. *Some Properties of Logarithmic Potential Functions.*

Analytic Quadrilaterals.—In the following pages we have frequently to do with simply connected finite regions bounded by four arcs, each of which is analytic inclusive of its extremities, the angles at the vertices being each greater than 0 and less than 2π. Such a region we will call an *analytic quadrilateral.*

LEMMA 1. *Let S be an analytic quadrilateral with sides C_1, C_1', C_2, C_2'. Then there exists a function u harmonic within S, equal to a_1 on C_1 and a_2 on C_2, a_1 and a_2 being constants, and having its inner normal derivative zero at all points of C_1' and C_2' except the extremities. If u is finite in S, u is unique.**

Let S be mapped conformal y on the upper half-plane, and hence, by an elliptic integral of the first kind, on a rectangle R in the $w = \xi + \eta i$-plane with its vertices at $\pm K$, $\pm K + K' i$, C_2 going into a segment of the axis of reals. Then the function

$$v = \frac{a_1 - a_2}{K'}\eta + a_2$$

fulfills the conditions of the theorem in R.

Let u be the function in S into which v is carried by the inverse transformations. Then u fulfills the conditions of the theorem in S.

* The first part of this theorem is not new. A rigorous proof of the uniqueness is, however, less familiar.

To show that u is unique, let u_1 be a second such function, and form the function $w = u - u_1$. Here, w is finite and harmonic within S, and

$$w|_{C_1} = 0, \qquad w|_{C_2} = 0, \qquad \left.\frac{\partial w}{\partial n}\right|_{C_1'} = 0, \qquad \left.\frac{\partial w}{\partial n}\right|_{C_2'} = 0.$$

From the lemma of § 2 it follows that w is a function to which Formula (B) of § 1 applies:

$$\int_S\int \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dS = -\int_C w\frac{\partial w}{\partial n}\,ds.$$

Since the line integral vanishes, the double integral must also vanish, and from this it follows that

$$w = \text{const.} = 0.$$

LEMMA 2. *Let S_1 be an analytic quadrilateral with the sides C_1, C_1', C_2, C_2', and let the function u_1 satisfy the following conditions*:

1) *u_1 is finite and harmonic within S_1;*

2) $u_1|_{C_1} = k_1, \quad u_1|_{C_2} = k_2,$

k_1 and k_2 being constants, and $k_1 > k_2 \geqq 0$;

3) $\left.\dfrac{\partial u_1}{\partial n}\right|_{C_1'} = 0, \qquad \left.\dfrac{\partial u_1}{\partial n}\right|_{C_2'} = 0.$

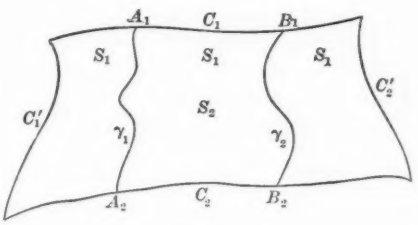


FIG. 4.

Let S_2 be a second analytic quadrilateral lying in S_1 and having two of its sides, A_1B_1 and A_2B_2, consisting of arcs of C_1 and C_2 respectively. If u_2 satisfies the same conditions in S_2 that u_1 does in S_1, then

$$-\int_{A_1B_1}\frac{\partial u_2}{\partial n}\,ds < -\int_{C_1}\frac{\partial u_1}{\partial n}\,ds.$$

From the lemma of § 2, and § 4, end, it follows that u_1 and u_2 are functions to which the formulas of § 1 are applicable. Hence

$$\int_{S_1}\int \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2\right] dS = -\int_{\sigma_1} u_1\frac{\partial u_1}{\partial n}\,ds,$$

where σ_1 denotes the complete boundary of S_1. Moreover, by § 1, (D),

$$\int_{\sigma_1}\frac{\partial u_1}{\partial n}\,ds = 0,$$

and since

$$\int_{\sigma_1} \frac{\partial u_1}{\partial n} ds = \int_{C_1} \frac{\partial u_1}{\partial n} ds + \int_{C_2} \frac{\partial u_1}{\partial n} ds,$$

it follows that

$$\int_{C_1} \frac{\partial u_1}{\partial n} ds = -\int_{C_2} \frac{\partial u_1}{\partial n} ds.$$

We thus obtain for the value of the double integral:

$$(9) \quad \iint_{S_1} \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2 \right] dS = -k_1 \int_{C_1} \frac{\partial u_1}{\partial n} ds - k_2 \int_{C_2} \frac{\partial u_1}{\partial n} ds = -(k_1 - k_2) \int_{C_1} \frac{\partial u_1}{\partial n} ds.$$

Similarly,

$$(10) \quad \iint_{S_2} \left[\left(\frac{\partial u_2}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial y}\right)^2 \right] dS = -(k_1 - k_2) \int_{A_1 B_1} \frac{\partial u_2}{\partial n} ds.$$

Let $w = u_1 - u_2$ in S_2; $u_1 = u_2 + w$. Then

$$\iint_{S_2} \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2 \right] dS = \iint_{S_2} \left[\left(\frac{\partial u_2}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right] dS + 2 \iint_{S_2} \left(\frac{\partial u_2}{\partial x} \frac{\partial w}{\partial x} + \frac{\partial u_2}{\partial y} \frac{\partial w}{\partial y} \right) dS.$$

The last integral, being equal to

$$-\int_{\sigma_2} w \frac{\partial u_2}{\partial n} ds,$$

is seen to vanish. Hence

$$\iint_{S_2} \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2 \right] dS \geqq \iint_{S_2} \left[\left(\frac{\partial u_2}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial y}\right)^2 \right] dS.$$

But

$$(11) \quad \iint_{S_1} \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2 \right] dS > \iint_{S_2} \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2 \right] dS.$$

Substituting for the integrals in (11) their values obtained from (9) and (10), we have

$$-(k_1 - k_2) \int_{C_1} \frac{\partial u_1}{\partial n} ds > -(k_1 - k_2) \int_{A_1 B_1} \frac{\partial u_2}{\partial n} ds,$$

and the proposition is proved.*

* The proof here given is due to Professor B. O. PEIRCE.

LEMMA 3. *In the analytic quadrilateral S with sides C_1, C_1', C_2, C_2' the functions u_1, u_2 shall satisfy the following conditions:*

1) *u_i is finite and harmonic within S,* $= 1, 2;$

2) $$\left.\frac{\partial u_i}{\partial n}\right|_{C_1'} = 0, \qquad \left.\frac{\partial u_i}{\partial n}\right|_{C_2'} = 0, \qquad (i = 1, 2);$$

3_1) $$0 < u_1 |_{C_1} < k, \qquad 0 < u_1 |_{C_2};$$

where k is a positive constant, and u_1 is harmonic in all points of C_1 and C_2;†

3_2) $$u_2 |_{C_1} = k, \qquad u_2 |_{C_2} = 0.$$

Then

$$-\int_{C_1} \frac{\partial u_1}{\partial n}\, ds < -\int_{C_1} \frac{\partial u_2}{\partial n}\, ds.$$

Let S be mapped on a rectangle R of the (x', y')-plane, C_1 going over into a segment AB of the line $y' = 1$, and C_2 into a segment CD of the axis of x'. Let l be the length of either of these segments. Denote by u_1', u_2' the functions into which u_1, u_2 are thus transformed. Then

$$u_2' = ky'.$$

For, ky' is evidently one function having the properties of u_2', and by Lemma 1 there is only one such function. We wish, then, to prove that

$$-\int_{AB} \frac{\partial u_2'}{\partial n'}\, ds' > -\int_{AB} \frac{\partial u_1'}{\partial n'}\, ds'. \tag{12}$$

Since

$$\left.\frac{\partial u_2'}{\partial n'}\right|_{AB} = -k, \qquad \left.\frac{\partial u_2'}{\partial n'}\right|_{CD} = k,$$

the inequality (12) is equivalent to the following:

$$\int_{AB} \frac{\partial u_1'}{\partial n'}\, ds' > -kl. \tag{13}$$

From § 1, (*C*), we have, since u_1' and u_2' satisfy in R the requisite conditions:

$$\int_\sigma u_2' \frac{\partial u_1'}{\partial n'}\, ds' = \int_\sigma u_1' \frac{\partial u_2'}{\partial n'}\, ds',$$

where σ denotes the complete boundary of R. Here,

$$\int_\sigma u_2' \frac{\partial u_1'}{\partial n'}\, ds' = \int_{AB} u_2' \frac{\partial u_1'}{\partial n'}\, ds' = k \int_{AB} \frac{\partial u_1'}{\partial n'}\, ds';$$

† This latter condition can be replaced by a less restrictive one; for example, that the first partial derivatives of u_1 be continuous on the boundary.

and

$$\int_{\sigma} u_1' \frac{\partial u_2'}{\partial n'} ds' = \int_{AB} u_1' (-k)\, ds' + \int_{CD} u_1' k\, ds' = k\left(- \int_{AB} u_1' ds' + \int_{CD} u_1' ds' \right).$$

Hence

$$\int_{AB} \frac{\partial u_1'}{\partial n'} ds' = - \int_{AB} u_1' ds' + \int_{CD} u_1' ds' > - \int_{AB} u_1' ds' > - kl,$$

and this is precisely (13).

§ 6. *The Fundamental Theorem and the Method of Proof.*

Definition: A boundary point A of a region S shall be said to be *accessible* if it can be approached along a Jordan curve lying wholly within the region except for its extremity A.* For a discussion of multiply counting boundary points cf. § 7, under Theorem III.

THEOREM I. *Let S be a simply connected plane region whose boundary consists of more than one point, and let the interior of S be mapped conformally on the interior of a circle S'. Let A be an accessible boundary point of S, and let C be a curve lying within S (except for one extremity) and leading to A. Then the image of C in S' is a curve C' with a single limiting point A' on the circumference of S'; so that, if a point P approach A along C, its image P' will approach A' as its limit.*

The region S can be transformed on a finite region lying in the upper half of the $z = x + yi$-plane, the point A going into the origin.† It is this latter region which we shall henceforth consider and denote by S. Let S' be the unit circle of the $w = u + vi$-plane, the point O of S going over into $w = 0$, and let g be the Green's function of S with its pole at O; h, the conjugate function. Then the interior of S is mapped conformally on the interior of S' by the transformation

$$w = e^{-g-hi}.$$

As P approaches A along C, g approaches 0. To prove the theorem, it is necessary and sufficient to prove that, as P approaches A, a branch of h, single-valued and continuous along C, also approaches a limit.

Let ϵ be a positive quantity chosen at pleasure and then held fast. With A as center construct two circumferences

$$\sigma_1 : \quad |z| = r_1; \qquad \sigma_2 : \quad |z| = r_2; \qquad \frac{r_1}{r_2} = a < 1,$$

* For an example of a region having boundary points that are not accessible cf. OSGOOD, these Transactions, vol. 1 (1900), p. 311.

† OSGOOD, *Funktionentheorie*, vol. 1, p. 700.

a being a positive constant less than unity. Furthermore, r_2 shall be so chosen that O shall lie outside the circle σ_2. Let P_1 be that point of intersection of σ_1 and C such that σ_1 and the arc P_1A of C have no other point in common. It will now be shown that r_2 can be so chosen that the variation of h along C from P_1 to A is less than ϵ.

Let η be a positive quantity chosen arbitrarily small. Then it is clear that r_2 can be so restricted that the upper limit of g on the arc or arcs of σ_2 that lie in S will be less than η. Let this be done. r_2, and with it r_1, are now fixed.

Let P' be a point on the arc P_1A arbitrarily near A, and let $\mu > 0$ be so chosen that every point of the arc P_1P' lies within the curve $g = \mu$. Denote by $\bar{S}$ the portion of S composed of the curve $g = \mu$ and the points interior to

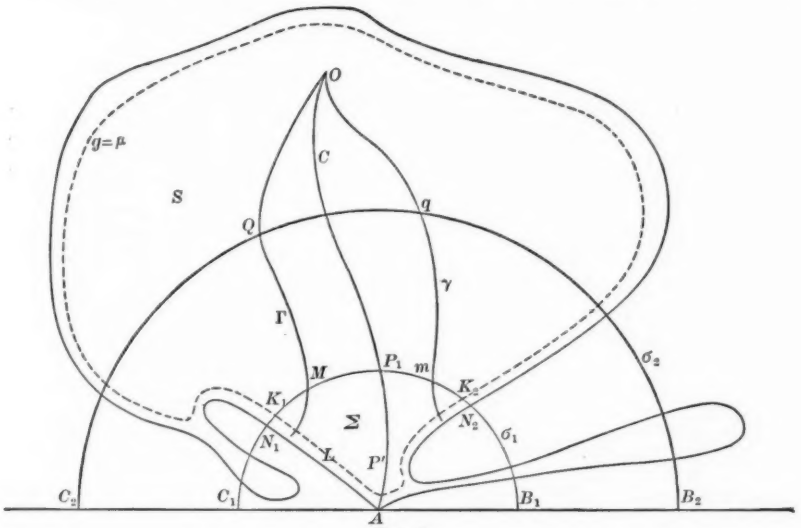


FIG. 5.

it. Since $g = \mu$ is a closed analytic curve distinct from σ_1, it is cut by σ_1 in a finite number of points. Thus σ_1 yields a finite number of cross-cuts of $\bar{S}$. One of these, K_1K_2, divides $\bar{S}$ into two simply connected regions, one of which contains the point O, and the other, $-\Sigma$, let us call it,—the point P' (and hence the arc P_1P' of C).

Since a branch of h is single-valued and harmonic in Σ and continuous on the boundary, it has an upper and a lower limit in Σ, which are reached on the boundary of Σ. Now $\partial g / \partial n > 0$ at every point of the curve $g = \mu$, and so h varies monotonically along the arc K_1LK_2. Therefore the upper limit H and the lower limit h_0 of h in Σ are reached on the arc K_1K_2 of σ_1.

Construct the curves $h = H$ and $h = h_0$, and denote them respectively

by Γ and γ. Let be M and Q points in which Γ meets σ_1 and σ_2 respectively, such that the arc MQ of Γ has no point except its extremities in common with σ_1 or σ_2. Let m and q have a corresponding meaning for γ. The points m, M may lie on the arc K_1K_2 produced.

The proof turns on applying Lemmas 2 and 3 of § 5 to the analytic quadrilaterals $B_1P_1C_1C_2QB_2$ and $mMQq$ (or a corresponding one, to be determined later), and noting at the outset the following two facts:

a) The value of g on the arc qQ is less than η;

b)
$$-\int_{mM}\frac{\partial g}{\partial n}ds = H - h_0.$$

The truth of (*b*) depends essentially on the fact that when a point P, starting at K_1, describes the boundary of Σ in the positive sense (K_1LK_2), h attains its minimum value h_0 along the arc $K_2P_1K_1$ in a point or in several points, all of which are reached *before* h attains its maximum value H on that arc. For a proof, consider the map of $\bar{S}$ on the circle $\rho \leqq e^{-\mu}$, where $w = \rho e^{\theta i}$, and the image of the arc $K_2P_1K_1$ in this circle. Since $h = -\theta + \text{const.}$, it is sufficient to prove the corresponding statement for θ, and this is readily done by mapping the above circle on the W-plane by the function $W = \log w$.

Lemma 2 is applied as follows: Form a function u_1 harmonic in the ring $r_1 < |z| < r_2$ and such that

$$u_1|_{\sigma_1} = 0, \qquad u_1|_{\sigma_2} = \eta.$$

It is

$$u_1 = \frac{\eta(\log r - \log r_1)}{\log r_2 - \log r_1} = c\eta(\log r - \log r_1),$$

where $c = -1/\log a$. The conjugate of u_1 is

$$v_1 = c\eta\varphi + \text{const.},$$

and

(14)
$$-\int_{B_2C_2}\frac{\partial u_1}{\partial n}ds = c\pi\eta.$$

Let u_2 be determined by the following conditions:

1) u_2 is finite and harmonic within the analytic quadrilateral $mMQq$,—we assume for the moment that γ and Γ are not tangent to σ_1 or σ_2;

2)
$$u_2|_{qQ} = \eta, \qquad u_2|_{mM} = 0;$$

3)
$$\left.\frac{\partial u_2}{\partial n}\right|_{\Gamma} = 0, \qquad \left.\frac{\partial u_2}{\partial n}\right|_{\gamma} = 0.$$

The existence and uniqueness of such a function follow from Lemma 1, § 5.

We now apply Lemma 2 of § 5, taking for the region S_1 so much of the ring

$r_1 \leqq |z| \leqq r_2$ as lies above the lower half-plane, C_1 lying along σ_2; and as S_2, the analytic quadrilateral $mMQq$. Hence, with the aid of (14),

$$-\int_{qQ} \frac{\partial u_2}{\partial n} ds < -\int_{B_2C_2} \frac{\partial u_1}{\partial n} ds = c\pi\eta. \tag{15}$$

Furthermore, from § 1, (*D*),

$$\int_{mMQq} \frac{\partial g}{\partial n} ds = 0,$$

and hence, by using (*b*), we have

$$-\int_{qQ} \frac{\partial g}{\partial n} ds = \int_{mM} \frac{\partial g}{\partial n} ds = H - h_0, \tag{16}$$

the normal in the last integral being opposite in sense to that of the integral in (*b*).

Finally, we apply Lemma 3 to the functions g and u_2 considered in $mMQq$, g corresponding to u_1. Hence

$$-\int_{qQ} \frac{\partial g}{\partial n} ds < -\int_{qQ} \frac{\partial u_2}{\partial n} ds,$$

or, from (15) and (16),

$$H - h_0 < c\pi\eta.$$

If, then, we choose η so that $2c\pi\eta < \epsilon$,—the reason for the factor 2 will appear presently,—our proof is complete, except for the assumption that Γ and γ are not tangent to σ_1 or σ_2.

In case of tangency, Γ and γ can be replaced by curves near by, cutting σ_1 and σ_2, but not tangent: $h = H'$ and $h = h_0'$, H' and h_0' differing but slightly from H and h_0 respectively; so that, in particular, $H' - h_0' > \frac{1}{2}(H - h_0)$. The point m', as above in the case of m and M, will thus be reached before M', so that (*b*) will hold. Using the analytic quadrilateral bounded in part by these curves instead of Γ and γ, we infer that

$$H' - h_0' < c\pi\eta, \qquad \text{or} \qquad H - h_0 < 2c\pi\eta < \epsilon.$$

This completes the proof.

Let the limit approached by h as P approaches A along C be denoted by $\bar{h}$.

Extension of Theorem I.

The analysis by which the foregoing theorem has been established yields more than the result stated in that theorem. Extend the arc mM of σ_1 in both directions until it meets the boundary of S in the points N_1 and N_2. The cross-cut $N_1P_1N_2$ divides S into two simply connected regions. Let Σ^0 be the one of these regions that contains the arc P_1A of C.

From the theorem just proved it follows that the function h approaches a limit when P approaches N_1, and also when P approaches N_2, along the arc mM produced. h has, therefore, a finite upper limit $\bar{H}$ and a finite lower limit $\bar{h}_0$ along the cross-cut. Moreover, it is readily seen from the foregoing analysis that

$$\bar{H} - \bar{h}_0 \leqq \epsilon;$$

and since, at any interior point of Σ^0, h lies between $\bar{H}$ and $\bar{h}_0$, we have, for any such point

$$| h - \bar{h} | \leqq \epsilon. \tag{17}$$

Consider, now, a succession of circles with A as center and with radii

$$r^0 = r_1 > r' > r'' > \cdots, \qquad \lim_{n=\infty} r^{(n)} = 0,$$

and the corresponding regions cut from S:

$$\Sigma^0, \quad \Sigma', \quad \Sigma'', \quad \cdots,$$

the general region being so taken as to include the whole of a certain arc of C abutting on A. Let $z_1, z_2, \cdots$ be any set of points interior to S with A as a cluster-point and such that, for a given m arbitrarily large, z_n lies in $\Sigma^{(m)}$ provided $n > n_m$. If h_n denotes the value of h in z_n, then h_n will approach a limit, and

$$\lim_{n=\infty} h_n = \bar{h}. \tag{18}$$

We have, therefore, obtained the following

EXTENSION OF THEOREM I. *If $w_1, w_2, \cdots$ be the images in S' of $z_1, z_2, \cdots$, and if $\bar{w}$ correspond to A', then w_n will approach a limit and*

$$\lim_{n=\infty} w_n = \bar{w}. \tag{19}$$

It is to be noted that the maximum diameter of $\Sigma^{(n)}$ does not necessarily converge toward zero when $n = \infty$, and also that the points z_n may have other cluster-points than A,—which points will, of course, lie on the boundary of S.

COROLLARY. *A point A' of the circumference of S' which is the image of an accessible boundary point A of S has other such points in every neighborhood of itself and on either side; i. e., the points A' form a set that is "in sich dicht".*

In fact, the extremities of the cross-cuts $| z | = r^{(n)}$ just described are accessible points of the boundary of S, which go over into points of the circumference of S' approaching A' from both sides.

The set of points A' is not, however, in general perfect, as is shown by the following example. Consider the region S_1 of the $z_1 = x_1 + iy_1$-plane:

$$S_1: \qquad 1 < x_1 \leqq 2, \qquad \frac{1}{x_1 - 1} \leqq y_1 \leqq \frac{1}{x_1 - 1} + 1,$$

and let a definite interior point O_1 be chosen as the pole of the Green's function for S_1. If S_1 be mapped on S_2 by the transformation

$$z_2 = \frac{1}{z_1},$$

then all the boundary points of S_2, inclusive of the cusp at the origin, are accessible. Let $h = h'$ be the line of flow to this point.

On the other hand, let S_1 be wrapped round a circle by the transformation

$$z_3 = e^{z_1}.$$

Then the curve $h = h'$ winds round this circle indefinitely, and its only points of condensation not pertaining to itself are the points of the circle. Let A be an arbitrary point of this circle, and let $z_1, z_2, \cdots$ be a set of points in S with A as their limiting point. Then their images w_n will converge toward a point A' on the boundary of S', and A' will be the same point for all of the above choices of A.

§ 7. *Further Theorems.*

THEOREM II. *Let C_1 be a second curve of S also leading from O to A and meeting C only in those points. Let C_1' be the image of C_1 in S', A_1' its limiting point on the boundary of S'. The necessary and sufficient condition that A_1' coincide with A' is that the simple closed curve $\bar{C}$ consisting of C and C_1 may be drawn together continuously to the point A without passing out of S; or, in other words, that $\bar{C}$ shall contain in its interior only interior points of S.*

That the condition is sufficient follows at once from the extension of Theorem I.

To prove the condition necessary, consider the simple closed curves $\bar{C}$ and $\bar{C}''$. Suppose $\bar{C}$ contains boundary points of S in its interior. Denote the part of the latter region lying within S by R, and consider the function w of z, defined by the map, in R. When z, remaining in R, approaches a boundary point B of S, passing through any set of points $z_1, z_2, \cdots$ which have B as their sole limiting point, w approaches a limit, namely, the value $\bar{w}$ of w in A'. By an extension of a theorem due to Painlevé* w must, therefore, reduce to a constant, $\bar{w}$, and this is not true.

Closely related to the foregoing theorem is the following.

* PAINLEVÉ, Toulouse Annales, vol. 2 (1888), p. B. 29, stated the following theorem: *If $f(z)$ is analytic in a region T and approaches one and the same value along an arc of the boundary, then $f(z)$ is a constant.* He proved the theorem for ordinary arcs. But it is true generally, when the words: "an arc of the boundary" are replaced by the words: "a connected piece of the boundary containing two distinct points, each of which can be approached along a curve lying within the region." Cf. TAYLOR, Bulletin of the American Mathematical Society, vol. 19, June, 1913.

THEOREM III. *If A and A_1 are two distinct accessible points of the boundary of S, then the corresponding points A' and A_1' on the circumference of S' will also be distinct.*

Conversely, let A' and A_1' be two distinct points of the circumference of S', corresponding to points A and A_1 (distinct or coincident) which can be reached along curves C and C_1; the latter moreover shall not intersect. Then each of the regions into which S is divided by the cross-cut (C, C_1) will contain among its boundary points other boundary points of S than A and A_1.

For, suppose A' and A_1' coincide. Then the reasoning in the proof of the last theorem is applicable, and it follows that A and A_1 coincide. Thus the first part of the proposition is established.

The second part of the theorem follows at once from the extension of Theorem I.

Theorem II may be regarded as coming directly under Theorem III if we agree to consider as distinct two coincident points of the boundary of S which can be approached along curves C and C_1 emanating from O and lying in S, when each of the regions into which S is divided by the cross-cut (C, C_1) has among its boundary points other boundary points of S than the given points. Such points of the boundary, if they exist, shall be called *multiple points.*

Again, a boundary point may be accessible for a certain avenue of approach, and still be inaccessible for a set of points lying within S and condensing on it as their sole limiting point. As an example of such a region, take the interior points of the circle $|z| < 5$ with the exception of those points (a) which lie on the line $x = 0, -1 \leqq y \leqq 1$, and (b) which lie on the curve

$$y = \sin \frac{1}{x}, \qquad 0 < x < x_0,$$

where x_0 is the smallest positive x for which this curve cuts the circumference of the circle.

It is, then, in general impossible to divide the boundary points into accessible and inaccessible points except as they are multiply counted. We will not attempt such a classification, but refer to Condition A below as giving a working definition for Theorem V.

COROLLARY 1. *If the boundary of S consist of a simple closed Jordan curve, then the conformal map of the interior of S on the interior of the circle S' will be one-to-one and continuous on the boundary.*

From Theorems II and III it follows that a one-to-one relation exists between the totality of points of the boundary of S and a set of points (not as yet known to be everywhere dense) on the circumference of S'. It follows further,

however, from the extension of Theorem I that the points of the latter set depend continuously on the points of the former set. And so Hurwitz's condition that the points of the latter set form a Jordan curve is seen to be fulfilled.*

COROLLARY 2: *If the boundary has no inaccessible and no multiple points, it is a simple closed Jordan curve.*

First, there will be a one-to-one relation between the points of the boundary of S and those of a set situated on the circumference of S'.

Secondly, the latter points must comprise all the points of that circumference. For otherwise consider a radius leading to an omitted point. The image of this radius in S would be a curve having more than one point of condensation on the boundary, and hence entering every neighborhood of two accessible points. But this is absurd.

Thirdly, if A' is an arbitrary point of the circumference of S' and A its image on the boundary of S; and if a circle of arbitrarily small radius be described about A, then there exists an arc extending equal distances to either side of A', all of whose points have their images in the circle in question. For otherwise there would exist a set of boundary, and therefore also a set of interior points of S, each of whose distances from A is greater than a certain positive constant, and such that their images in S' cluster about A'. But these points would have a cluster point A_1 on the boundary of S distinct from A, and hence go over into points of S' condensing on A_1'.

Thus the boundary points of S fulfil Hurwitz's definition of a Jordan curve, and the theorem is proved.

Remark: It may however happen, even when the boundary has inaccessible points, that the accessible points are related in a one-to-one manner to the points of the circumference of S'. An example of such a region is the domain considered by Osgood, *Funktionentheorie*, vol. 1, p. 154.

This example shows, moreover, that Painlevé's theorem cannot be extended as follows: Let $f(z)$ be analytic in a region T and vanish along a portion of the boundary consisting of an arc of an analytic curve; then $f(z) \equiv 0$. For the foregoing function w approaches one and the same value, w_0, along the right line segment $x = 0, 0 \leqq y \leqq 1$. But $w - w_0 \not\equiv 0$.

THEOREM IV. *The points A' are everywhere dense on the circumference of S'.*

Suppose the proposition were not true. Let $h = -\theta$, and let the arc $-L \leqq \theta \leqq -l$ lie within an arc free from points A'. The region S will be divided by the curves $h = l$ and $h = L$ into two regions, one of which, S_1, —namely, that in which $l < h < L$,—has no accessible boundary points except those of the curves $h = l$ and $h = L$.

* A. HURWITZ, *Verhandlungen des ersten Internationalen Mathematikerkongresses*, 1898, p. 102; OSGOOD, *Funktionentheorie*, vol. 1, p. 147.

Consider the curve $h = l_1 = \frac{1}{2}(l + L)$, and let A be a point of condensation of this curve not pertaining to the curve. Then A will be a boundary point of S. Let the plane be transformed as in the proof of Theorem I so that A will come to the origin and S lie in the upper half-plane; S and A now referring to the transformed region and point.

Choose the positive quantity η arbitrarily small and draw the circles σ_1 and σ_2 as before, the Green's function g of S being less than η wherever it is defined on and inside of σ_2, and $r_1 / r_2 = a < 1$.

The curve $h = l_1$ must enter the inner circle. Let P_1 be a point in which it meets that circle. Then an arc of the circle, emanating from P_1 and remaining in S_1, will approach the boundary of S_1 in a point P_2 lying on one of the curves $h = l$, $h = L$; and the arc P_1P_2 will lie within S. The variation in h along this arc is $\frac{1}{2}(L - l)$.

Choose $\mu > 0$ so that the curve $g = \mu$ includes the arc P_1P_2 in its interior, and denote the points of intersection of this arc produced with $g = \mu$ by K_1 and K_2. We have now a region Σ cut off from $\bar{S}$ (the interior and boundary of $g = \mu$) by the arc K_1K_2 precisely like the region Σ of the proof of Theorem I. Let H and h_0 be the maximum and minimum values of h in Σ. They are attained on the arc K_1K_2 of the boundary. And now the earlier reasoning leads here, as in that case, to the conclusion that

$$H - h_0 < 2\pi c\eta .$$

But $H - h_0$ is at least as great as $\frac{1}{2}(L - l)$, and η may be so chosen that $2\pi c\eta < \frac{1}{2}(L - l)$. In this contradiction lies the proof of the theorem.

We are now in a position to state a necessary and sufficient condition that a curve C of S, any complete arc of which is a Jordan curve and which proceeds out toward the boundary of S, have as its image in S' a curve C' abutting on a single boundary point of S'.

Condition A. It may happen that there are two distinct accessible boundary points A and A_1 of S and neighborhoods

$$\Sigma^0, \Sigma', \Sigma'', \cdots, \qquad \Sigma_1^0, \Sigma_1', \Sigma_1'', \cdots$$

as described under the extension of Theorem I such that each $\Sigma^{(n)}$ and each $\Sigma_1^{(n)}$ is traversed by C. If this is not the case, we say that C fulfills Condition A.

THEOREM V. *A necessary and sufficient condition that the image C' of the above curve C abut on a single boundary point of S' is that C fulfil Condition A.*

That the condition is necessary, is obvious. To show that it is sufficient, suppose B and B_1 were two distinct boundary points of S' which are points of

condensation of C'. Since C', as it advances, ultimately has its later points all outside of any circle concentric with S' and lying within it, at least one of the arcs BB_1 must have all of its points points of condensation of C'. But the points A' which are the images of accessible boundary points A are by Theorem IV everywhere dense along the circumference of S'. Hence there must be two distinct accessible points of the boundary of S precluded by the hypothesis, and the proposition is established.

COROLLARY. *The image of a curve C' of S' having but a single point of condensation on the circumference of S' is a curve C satisfying Condition A.*

§ 8. *Multiply Connected and Multiple-Leaved Regions.*

The foregoing results apply directly to the case of an arbitrary region S, single or multiple-leaved, simply or multiply connected, and bounded in part by a set of points $\mathfrak{Q}$ more than one in number and such that two of these points, A and A_1, can be connected by a cross-cut C cutting off from the region a simply connected piece S_1 bounded completely by C and $\mathfrak{Q}$. If S be so mapped that S_1 goes over into a region S_1' bounded in part by an analytic curve $\mathfrak{Q}'$ corresponding to $\mathfrak{Q}$, then the same conditions for continuity on the boundary of the map exist as in the case hitherto discussed.

For, the interior of S_1 can be mapped conformally on the interior of a circle $\bar{S}_1'$, the part of S_1 abutting on $\mathfrak{Q}$ going over into a part of $\bar{S}_1'$ abutting on an arc $\bar{\mathfrak{Q}}_1'$ of the circumference. To the map in the neighborhood of $\bar{\mathfrak{Q}}_1'$ the earlier results apply. On the other hand, the complete neighborhood of any point of $\bar{\mathfrak{Q}}_1'$ (exclusive of the extremities of the arc) is mapped conformally on the complete neighborhood of the corresponding point of $\mathfrak{Q}'$.

As an example, consider a plane region S with a boundary of n pieces, each piece consisting of more than a single point. This region can first be transformed into a finite region by considering the part of the whole plane bounded by a single piece and having an interior point lying within S. This latter region can then be transformed on the interior of a circle, and thus the given region goes over into a region lying within a circle and having the circle as a piece of the boundary.

Proceed now in a similar manner with a piece of the boundary of the new region lying inside the circle. Thus again a region is obtained lying within a circle and having the circle as one piece of its boundary, a second piece of its boundary being now a simple closed analytic curve.

By repeating the process we finally obtain a region $\mathfrak{S}$ whose boundary consists of n simple closed analytic curves not cutting one another. The map on the boundary obeys the same laws as in the case of the simply connected plane region discussed in the foregoing paragraphs. Thus a curve of S abutting

on an accessible point of the boundary goes over into a curve of $\mathfrak{S}$ having a single point of condensation on the boundary.*

Or, again, since $\mathfrak{S}$ can be mapped on an n-sheeted Riemann's surface spread out over the upper half-plane, its leaves being connected by $2n - 2$ branch points, it follows in particular that S can be mapped on such a surface, and the map on the boundary (the axis of reals) will obey the laws obtained in the preceding paragraphs.

*It is only the part of this theorem that relates to the map on the boundary which is new. The part relating to the map in the interior was obtained by the authors prior to June, 1906, but was not published by them. It has since appeared in the literature.

HARVARD UNIVERSITY,
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